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Characterizing Heavy-Tailed Distributions Induced by Retransmissions

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Abstract

Consider a generic data unit of random size L that needs to be transmitted over a channel of unit capacity. The channel availability dynamics is modeled as an i.i.d. sequence $\{A, A_i\}_{i\geqslant 1}$ that is independent of L. During each period of time that the channel becomes available, say A_i, we attempt to transmit the data unit. If $L \leq A_i$, the transmission is considered successful; otherwise, we wait for the next available period A_{i+1} and attempt to retransmit the data from the beginning. We investigate the asymptotic properties of the number of retransmissions N and the total transmission time T until the data is successfully transmitted. In the context of studying the completion times in systems with failures where jobs restart from the beginning, it was first recognized in [5, 18] that this model results in power law and, in general, heavy-tailed delays. The main objective of this paper is to uncover the detailed structure of this class of heavy-tailed distributions induced by retransmissions.

More precisely, we study how the functional dependence $(\mathbb{P}[L > x])^{-1} \approx \Phi((\mathbb{P}[A > x])^{-1})$ impacts the distributions of N and T; the approximation $\approx$ will be appropriately defined in the paper depending on the context. In the functional space of $\Phi(\cdot)$, we discover several functional criticality points that separate classes of different functional behavior of the distribution of N. For example, we show that if $\log(\Phi(n))$ is slowly varying, then $\log(\mathbb{P}[N > n])$ is essentially slowly varying as well. Interestingly, if $\log(\Phi(n))$ grows slower than $e^{\sqrt{\log n}}$ then we have the asymptotic equivalence $\log(\mathbb{P}[N > n]) \approx -\log(\Phi(n))$. However, if $\log(\Phi(n))$ grows faster than $e^{\sqrt{\log n}}$, this asymptotic equivalence does not hold and admits a different functional form. Similarly, different types of functional behavior are shown for moderately heavy tails (Weibull distributions) where $\log(\mathbb{P}[N > n]) \approx -(\log \Phi(n))^{1/(\beta+1)}$ assuming $\log \Phi(n) \approx n^{\beta}$, as well as the nearly exponential ones of the form $\log(\mathbb{P}[N > n]) \approx -n/(\log n)^{1/\gamma}, \gamma > 0$ when $\Phi(\cdot)$ grows faster than two exponential scales $\log\log(\Phi(n)) \approx n^{\gamma}$.

We also discuss the engineering implications of our results on communication networks since retransmission strategy is a fundamental component of the existing network protocols on all communication layers, from the physical to the application one.

Keywords: Retransmissions, Channel (systems) with failures, Restarts, Origins of heavy-tails (subexponentiality), Gaussian distributions, Exponential distributions, Weibull distributions, Log-normal distributions, Power laws.

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1 Introduction

Retransmissions represent one of the most fundamental approaches in communication networks that guarantee data delivery in the presence of channel failures. These types of mechanisms have been employed on all networking layers, including, for example, Automatic Repeat reQuest (ARQ) protocol (e.g., see Section 2.4 of [3]) in the data link layer where a packet is resent automatically in case of an error; contention based ALOHA type protocols in the medium access control (MAC) layer that use random backoff and retransmission mechanism to recover data from collisions; end-to-end acknowledgement for multi-hop transmissions in the transport layer; HTTP downloading scheme in the application layer, etc. We discuss the engineering implications of our results at the end of this introduction and, in more detail, in Section 3.

As briefly stated in the abstract, we use the following generic channel with failures [11] to model the preceding situations. The channel dynamics is described as an on-off process $\{(A, U), (A_i, U_i)\}_{i\geqslant 1}$ with alternating periods when channel is available A_i and unavailable U_i, respectively; $(A, A_i)_{i\geqslant 1}$ and $(U, U_i)_{i\geqslant 1}$ are two independent sequences of i.i.d random variables. In each period of time that the channel becomes available, say A_i, we attempt to transmit the data unit of random size L. If $L \leq A_i$, we say that the transmission is successful; otherwise, we wait for the next period A_{i+1} when the channel is available and attempt to retransmit the data from the beginning. We study the asymptotic properties of the distributions of the total transmission time T and number of retransmissions N, for the precise definitions of these variables and the model, see the following Subsection 1.1.

The preceding model was introduced and studied in [14] and, apart from the already mentioned applications in communications, it represents a generic model for other situations where jobs have to restart from the beginning after a failure. It was first recognized in [5] that this model results in power law distributions when the distributions of L and A have a matrix exponential representation, and this result was rigorously proved and further generalized in [18]. Under more general conditions, [11] discovers that the distributions of N and T follow power laws with the same exponent α as long as $\log \mathbb{P}[L > x] \approx \alpha \log \mathbb{P}[A > x]$ for large x, which implies that power law distributions, possibly with infinite mean ($0 < \alpha < 1$) and variance ($0 < \alpha < 2$), may arise even when transmitting superexponential (e.g., Gaussian) documents/packets. More recent results on the heavy-tailed completion times in a system with failures are developed in [2]. In this paper, we further characterize this class of heavy-tailed distributions that are induced by retransmissions.

Technically speaking, our proofs are based on the method introduced in [11] that uses the following key arguments. First, in exploring the distribution of N, we assume that the functional relationship $\Phi(\cdot)$, with $\bar{F}^{-1}(x) \approx \Phi(\bar{G}^{-1}(x))$ between the probability distributions of $\bar{F}(x) \triangleq \mathbb{P}[L > x]$ and $\bar{G}(x) \triangleq \mathbb{P}[A > x]$, is eventually monotonically increasing, which guarantees the existence of an asymptotic inverse $\Phi^{\leftarrow}(\cdot)$ of $\Phi(\cdot)$, and then, we use the result that $\bar{F}(L)$ is a uniform random variable on $(0, 1)$ given that $\bar{F}(\cdot)$ is continuous (see [11, 9]), e.g., for $\bar{F}(x) = \left(\bar{G}(x)\right)^{\alpha}, \alpha > 0$, the key argument on the uniform distribution of $\bar{F}(L)$ from [11] can be illustrated as

$$\mathbb{P}[N > n] = \mathbb{E}\left[(1 - \bar{G}(L))^n\right] \approx \mathbb{E}[e^{-n\bar{G}(L)}] = \mathbb{E}\left[e^{-n\bar{F}^{1/\alpha}(L)}\right] = \frac{\Gamma(\alpha + 1)}{n^{\alpha}}.$$

Second, in contrast to [18, 2], instead of studying the total transmission time T directly, we study a simpler quantity N and then use the large deviations technique to investigate T, since T can be represented as a sum of L and $\{(A_i + U_i)\}_{1\leqslant i<N}$; see equation (1.1) in the next subsection. Hence, our analysis is entirely probabilistic, which differs from the work in [2] that relies on Tauberian theorems.

More precisely, we extend the results from [2, 11] under a more unified framework and study how the functional dependence between the data characteristics and channel dynamics in the form $(\mathbb{P}[L > x])^{-1} \approx \Phi(\mathbb{P}[A > x])^{-1})$ impacts the distribution of N, where the approximation $\approx$ will be possibly differently defined according to the context. In the functional space of $\Phi(n)$, we identify several functional criticality points that define different classes of functional behavior of the distribution of N. Specifically, in Subsection 2.1.1, we show that if $\Phi(n)$ is dominantly varying, e.g., regularly varying, then $\mathbb{P}[N > n] \approx \Phi(n)^{-1}$; see Proposition 2.2 and Theorem 2.1. As shown in Proposition 2.3, the preceding tail equivalence between $\mathbb{P}[N > n]$ and $\Phi(n)^{-1}$ basically does not hold if $\Phi(x)$ is not dominantly varying, e.g., if $\Phi(x)$ is lognormal. Furthermore, we show in a weaker form that if $\log(\Phi(n))$ is slowly varying, then $\log\left((\mathbb{P}[N > n])^{-1}\right)$ is essentially slowly varying as well, as proved in Proposition 2.1. Interestingly, if $\log(\Phi(n))$ grows slower than $e^{\sqrt{\log n}}$ then we have the asymptotic equivalence $\log(\mathbb{P}[N > n]) \approx -\log(\Phi(n))$ as shown in Theorem 2.2 and Corollary 2.3, which implies parts (1:1), (2:1) and (2:2) of Theorem 2.1 in [2] and extends Theorem 2 in [11]. However, if $\log(\Phi(n))$ grows faster than $e^{\sqrt{\log n}}$, this asymptotic equivalence does not hold and we demonstrate a different functional form in Proposition 2.5.

Next, for lighter distributions of Weibull type, in Subsection 2.1.2, we show that if $\log(\Phi(n))$ is regularly varying with index $\beta > 0$, then basically one obtains Weibull distribution for N, i.e., $\log(\mathbb{P}[N > n]) \approx -(\log \Phi(n))^{1/(\beta+1)}$, as shown in Theorem 2.3, which we term moderately heavy (Weibull tail) asymptotics; this result implies part (1:2) of Theorem 2.1 in [2], and provides a more precise logarithmic asymptotics instead of a double logarithmic limit. Finally, in Subsection 2.1.3, we consider the situation when the separation between $\mathbb{P}[L > x]$ and $\mathbb{P}[A > x]$ is very large, i.e., their distributions are roughly separated by more than two exponential scales $(\log\log(\Phi(n)) \approx n^{\gamma})$. This separation results in what we call the nearly exponential distribution for N in the form $\log(\mathbb{P}[N > n]) \approx -n/(\log n)^{1/\gamma}$.

After the preceding characterization of the different classes of distributional behavior for N, we study in Subsection 2.2 the total transmission time T. As previously stated for studying T, we use the large deviation results since T can be represented as the sum of L and $\{(A_i + U_i\}_{1 \leqslant i < N}$. In this context, our primary results show that: (i) when $\Phi(\cdot)$ is regularly varying, we derive the exact asymptotics for T in Theorem 2.5. (ii) when $\log(\Phi(\cdot))$ is slowly varying, we obtain the logarithmic asymptotics for T in Theorem 2.6. (iii) when $\log(\Phi(\cdot))$ is regularly varying with positive index, we derive, in a different scale than in Theorem 2.6, the logarithmic asymptotics in Theorem 2.7. Note that the preceding three results on T correspond to Theorems 2.1 i), 2.2 and 2.3 on N, respectively. Similarly, one can derive the respective statements on $\mathbb{P}[T > t]$ for other results on $\mathbb{P}[N > n]$, but we omit this to avoid lengthy expositions and repetitions. Interestingly, we want to point out that, unlike Theorems 2.5 and 2.6 requiring no conditions on A (Theorem 2.5 needs $\mathbb{E}[A] < \infty$), the minimum conditions needed for Theorem 2.7, as shown by Proposition 2.7, basically involve a balance between the tail decays of $\mathbb{P}[A > x]$ and $\mathbb{P}[L > x]$.

From a practical perspective, our results suggest that careful examination and possible redesign of retransmission based protocols in communication networks might be needed. This is especially the case for Ad Hoc and resource limited sensor networks, where frequent channel failures occur due to a variety of reasons, including signal fading, multipath effects, interference, contention with other nodes, obstructions, node mobility, and other changes in the environment [16]. In engineering applications, our main discovery is the matching between the statistical characteristics of the channel and transmitted data (packets). On the network application layer, most of us have been inconvenienced when the connections would brake while we are downloading a large file from the Internet. This issue has been already recognized in

practice where software for downloading files was developed that would save the intermediate data (checkpoints) and resume the download from the point when the connection was broken. However, our results emphasize that, in the presence of frequently failing connections, the long delays may arise even when downloading relatively small documents. Hence, we argue that one might need to modify the application layer software, especially for the wireless environment, by introducing checkpoints even for small to moderate size documents. In our related papers, we found that several well-known retransmission based protocols in different layers of networking architecture can lead to power law delays, e.g., ALOHA type protocols in MAC layer [9] and end-to-end acknowledgements in transport layer [10]. These new findings suggest that special care should be taken when designing robust networking protocols, especially in the wireless environment where channel failures are frequent. We discuss these and other engineering implications of our results in Section 3.

We also discuss possible solutions to alleviate this problem, such as assigning checkpoints, breaking large packets into smaller units preferably by using dynamic packet fragmentation techniques [13]. Clearly, there is a tradeoff between the sizes of these newly created packets and the throughput since, if the packets are too small, they will mostly contain the packet headers and, thus, very little useful information.

Finally, we would like to point out that, in addition to the preceding applications in communication networks [10, 11, 9, 13] and job processing on machines with failures [5, 18], the model studied in this paper may represent a basis for understanding more complex failure prone systems, e.g., see the recent study on parallel computing in [1].

The rest of the paper is organized as follows. After a detailed description of the channel model in the next Subsection 1.1, we present our main results in Section 2 that is composed of two parts: the asymptotics of the distribution of N in Subsection 2.1 and the asymptotics of the distribution of T in Subsection 2.2. In Subsection 2.1 we study three types of distinct behavior, i.e., the very heavy asymptotics in Subsection 2.1.1, the medium heavy (Weibull) asymptotics in Subsection 2.1.2 and the nearly exponential asymptotics in Subsection 2.1.3. Then, we concludes the paper with engineering implications in Section 3, which is followed by Section 4 that contains some of the technical proofs that have been deferred from the preceding sections.

1.1 Description of the Channel

In this section, we formally describe our model and provide necessary definitions and notation. Consider transmitting a generic data unit of random size L over a channel with failures. Without loss of generality, we assume that the channel is of unit capacity. The channel dynamics is modeled as an on-off process $\{(A_i, U_i)\}_{i\geqslant 1}$ with alternating independent periods when channel is available A_i and unavailable U_i, respectively. In each period of time that the channel becomes available, say A_i, we attempt to transmit the data unit and, if $L \leq A_i$, we say that the transmission was successful; otherwise, we wait for the next period A_{i+1} when the channel is available and attempt to retransmit the data from the beginning. A sketch of the model depicting the system is drawn in Figure 1.

Assume that $\{U, U_i\}_{i\geqslant 1}$ and $\{A, A_i\}_{i\geqslant 1}$ are two mutually independent sequences of i.i.d. random variables.

Definition 1.1 *The total number of (re)transmissions for a generic data unit of length L is defined as*

$$N \triangleq \inf\{n : A_n \geqslant L\},$$

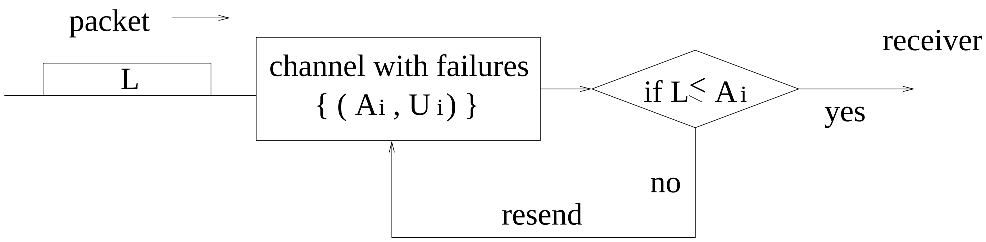

Figure 1: Packets sent over a channel with failures

and, the total transmission time for the data unit is defined as

$$T \triangleq \sum_{i=1}^{N-1} (A_i + U_i) + L. \tag{1.1}$$

We denote the complementary cumulative distribution functions for A and L, respectively, as

$$\bar{G}(x) \triangleq \mathbb{P}[A > x]$$

and

$$\bar{F}(x) \triangleq \mathbb{P}[L > x].$$

It was first discovered in Theorem 6 of [18] that this model leads to subexponential delay T under quite general conditions. The following slightly more general proposition was proven in Lemma 1 of [11] using probabilistic arguments (see also Proposition 1.2 in [2]).

Proposition 1.1 *If $\bar{F}(x) > 0$ for all $x \geqslant 0$, then both N and T are* subexponential *in the following sense that, for any $\epsilon > 0$,*

$$e^{\epsilon n}\mathbb{P}[N > n] \to \infty \quad \text{as } n \to \infty \tag{1.2}$$

and

$$e^{\epsilon t}\mathbb{P}[T > t] \to \infty \quad \text{as } t \to \infty. \tag{1.3}$$

Clearly, the preceding proposition defines a class of subexponential distributions that are induced by retransmissions; the **proof** of this proposition is presented in Subsection 4.1 for readers' convenience. The main study of this paper is to uncover the detailed structure of this class of distributions. More precisely, we investigate how the functional dependence of $\bar{F}$ and $\bar{G}$ (stated in the form $\bar{F}^{-1}(x) \approx \Phi(\bar{G}^{-1}(x))$) impacts the tail behavior of the distributions of both N and T, and the exact meaning of $\approx$ will be defined according to the context.

2 Main Results

This section presents our main results. Here, we assume that $\bar{F}(x)$ is a continuous function with support on $[0, \infty)$. If $\bar{F}(x)$ is lattice valued, our results may still hold; see Remarks 3 and 6. If $\bar{F}(x)$ has only a finite support, we discuss this situation in Section 3; see also Example 3 in Section IV of [11] and Section 3 of [2]. According to (1.1), the total transmission time T naturally depends on the number of transmissions N, and therefore, we first study the distributional properties of the number of transmissions N in Subsection 2.1, and then evaluate the total transmission time T using the large deviation approach in Subsection 2.2.

2.1 Asymptotics of the Distribution of the Number of Retransmissions N

This subsection presents the asymptotic results for the number of retransmissions N depending on the functional relationship $\Phi(\cdot)$ between $\bar{F}$ and $\bar{G}$. Informally, we study three scenarios: very heavy asymptotics (when $\log(\Phi(n))$ is slowly varying), medium heavy (Weibull) asymptotics (when $\log(\Phi(n))$ is regularly varying), and nearly exponential (when $\log\log(\Phi(n))$ is regularly varying), where within and between these subclasses we also identify critical functional points that define different distributional behavior of N.

More precisely, we show that:

1. If $\Phi(n)$ is dominantly regularly varying, e.g., regularly varying, then $\mathbb{P}[N > n] \approx \Phi(n)^{-1}$, as stated in Proposition 2.2 and Theorem 2.1.

2. If $\Phi(n)$ is not dominantly regularly varying, e.g., $\Phi(n)^{-1}$ being lognormal, the preceding tail equivalence $\mathbb{P}[N > n] \approx \Phi(n)^{-1}$ basically does not hold, as shown in Proposition 2.3. However, we show in a weaker form that, if $\log(\Phi(n))$ is slowly varying, then $\log(\mathbb{P}[N > n])$ is essentially slowly varying as well, as proved in Proposition 2.1. Interestingly, within this class, we discover two types of distinct functional behavior of $\log \mathbb{P}[N > n]$ depending on the growth of $\log(\Phi(n))$:

   (a) If $\log(\Phi(n))$ grows slower than $e^{\sqrt{\log n}}$, then we have the asymptotic equivalence $\log(\mathbb{P}[N > n]) \approx -\log(\Phi(n))$, as shown in Theorem 2.2.

   (b) This asymptotic equivalence does not hold if $\log(\Phi(n))$ grows faster than $e^{\sqrt{\log n}}$, and we demonstrate a different functional form in Proposition 2.5.

3. If $\log(\Phi(n))$ is regularly varying with index $\beta > 0$, then basically one obtains a Weibull distribution for N, $\log \mathbb{P}[N > n] \approx -(\log \Phi(n))^{1/(\beta+1)}$, as presented in Theorem 2.3.

4. When the decay of $\mathbb{P}[L > x]$ is much faster than $\mathbb{P}[A > x]$, i.e., $\log\log \mathbb{P}[L > x]^{-1} \approx R_\gamma(\log \mathbb{P}[A > x]^{-1})$ with $R_\gamma(\cdot), \gamma > 1$ being regularly varying, we obtain nearly exponential distributions for N in the form $\log(\mathbb{P}[N > n]) \approx n/R_\gamma^{\leftarrow}(\log n)$ with $R_\gamma^{\leftarrow}(n)$ being regularly varying with $0 < 1/\gamma < 1$, implying that $R_\gamma^{\leftarrow}(\log n)$ is slowly varying; see Theorem 2.4.

Our **proving method** is based on the following two key arguments:

1. $\Phi(x)$ is eventually monotonically increasing, which guarantees the existence of an inverse function $\Phi^{\leftarrow}(x)$ of $\Phi(x)$ when x is large enough.

2. $\bar{F}(x)$ is continuous, which implies that $V \triangleq \bar{F}(L)$ is a uniform random variable on $(0, 1)$, e.g., see Proposition 2.1 in Chapter 10 of [17]. Furthermore, our method essentially extends to lattice valued $\bar{F}(x)$ as well, as discussed in Remarks 3 and 6.

2.1.1 Very Heavy Asymptotics

This subsection studies the situation when the distribution of the number of retransmissions N is heavier than Weibull distributions. Specifically, we answer under what conditions $\mathbb{P}[N > n] \approx \Phi(n)^{-1}$ holds assuming $\bar{F}^{-1}(x) \approx \Phi(\bar{G}^{-1}(x))$, meaning that the complementary cumulative distribution function of N is of the same form (in terms of $\Phi(\cdot)$) as the functional relationship $\Phi(\cdot)$ between $\bar{F}$ and $\bar{G}$.

We term this subclass very heavy distributions since if $\log(\Phi(\cdot))$ is slowly varying, then the number of retransmissions N is always heavier than Weibull distribution, which is stated in the following Proposition 2.1.

Proposition 2.1 *If* $\log(\Phi(\cdot))$ *is slowly varying and*

$$\lim_{x\to\infty} \frac{\log\left(\bar{F}(x)^{-1}\right)}{\log\left(\Phi\left(\bar{G}(x)^{-1}\right)\right)} = 1, \tag{2.1}$$

then, for any $\epsilon > 0$*, as* $n \to \infty$*,*

$$\lim_{n\to\infty} \frac{\log\left(\mathbb{P}[N > n]^{-1}\right)}{n^{\epsilon}} = 0.$$

The **proof** of this proposition will be presented in Subsection 4.2. In the remainder of this subsection we study the detailed structure of this class of distributions that have very heavy tails. The Weibull distribution will be studied in the next Subsection 2.1.2 on medium heavy asymptotics.

Definition 2.1 *For an eventually non-decreasing function* $\Phi(x) : \mathbb{R}^+ \to \mathbb{R}^+$*, we say that* $\Phi(x)$ *is dominantly regularly varying if*

$$\overline{\lim_{x\to\infty}} \frac{\Phi(ex)}{\Phi(x)} < \infty, \tag{2.2}$$

where $e \equiv \exp(1)$*.*

In the paper we use the following standard notation. For any two real functions $a(t)$ and $b(t)$ and fixed $t_0 \in \mathbb{R}\cup\{\infty\}$, we use $a(t) \sim b(t)$ as $t \to t_0$ to denote $\lim_{t\to t_0}[a(t)/b(t)] = 1$. Similarly, we say that $a(t) \gtrsim b(t)$ as $t \to t_0$ if $\underline{\lim}_{t\to t_0} a(t)/b(t) \geq 1$; $a(t) \lesssim b(t)$ has a complementary definition. In addition, we say that $a(t) = o(b(t))$ as $t \to t_0$ if $\lim_{t\to t_0} a(t)/b(t) = 0$. When $t_0 = \infty$, we often simply write $a(t) = o(b(t))$ without explicitly stating $t \to \infty$ in order to simplify the notation. Also, we use the standard definition of an inverse function $f^{\leftarrow}(x) \triangleq \inf\{y : f(y) > x\}$ for a non-decreasing function $f(x)$; note that the notation $f^{-1}(x)$ is reserved for $1/f(x)$.

The following two propositions show that $\mathbb{P}[N > n]$ is tail equivalent to $\Phi(n)^{-1}$ basically only when $\Phi(n)$ is dominantly regularly varying.

Proposition 2.2 *If, as* $x \to \infty$*,*

$$\bar{F}^{-1}(x) \sim \Phi(\bar{G}^{-1}(x)), \tag{2.3}$$

then, there is finite $c \geqslant 1$ *such that*

$$c^{-1} \leqslant \underline{\lim_{n\to\infty}} \mathbb{P}[N > n]\Phi(n) \leqslant \overline{\lim_{n\to\infty}} \mathbb{P}[N > n]\Phi(n) \leqslant c.$$

Remark 1 Note that for this result as well as those in the rest of the paper we could have equivalently assumed that $\bar{F}(x) \sim \Phi(\bar{G}(x))$ where $\Phi(\cdot)$ is eventually non-increasing and satisfies the appropriate regularity conditions in the neighborhood of 0, e.g., condition (2.3) would be restated in the neighborhood of 0. In this case, the respective statement would be in the form $\mathbb{P}[N > n] \approx \Phi(n^{-1})$. Furthermore, the current form has additional notational benefits in the later sections, e.g., $\log\log\Phi(n)$ would need to be replaced by $\log\left(-\log(\Phi(n^{-1}))\right)$ in (say) Proposition 2.5.

Proposition 2.3 *If (2.3) is satisfied and* $\Phi(x)$ *is eventually non-decreasing with*

$$\lim_{x\to\infty} \frac{\Phi(ex)}{\Phi(x)} = \infty,$$

then,

$$\lim_{n\to\infty} \mathbb{P}[N > n]\Phi(n) = \infty.$$

When $\Phi(\cdot)$ is regularly varying, which is a subset of the dominantly regularly varying functions, we can compute the exact asymptotics of the distribution of N.

Theorem 2.1 *Assuming $\bar{F}^{-1}(x) \sim \Phi\left(\bar{G}^{-1}(x)\right)$ where $\Phi(\cdot)$ is regularly varying with index α, we obtain:*

i) If $\alpha > 0$, then, as $n \to \infty$,

$$\mathbb{P}[N > n] \sim \frac{\Gamma(\alpha+1)}{\Phi\left(n\right)}. \tag{2.4}$$

ii) If $\alpha = 0$ (meaning $\Phi(\cdot)$ is slowly varying) and $\Phi(x)$ is eventually non-decreasing, then, as $n \to \infty$,

$$\mathbb{P}[N > n] \sim \frac{1}{\Phi(n)}. \tag{2.5}$$

Remark 2 For $\alpha > 0$, this theorem was proved in Theorem 4 of [11] using the method that we further expand in this paper; alternatively, a similar result for T was proved using Tauberian method in Theorem 2.2 of [2]. We will prove the corresponding result for T in Theorem 2.5 in Subsection 2.2.

Remark 3 (Lattice variables) Note that if $\bar{F}(x)$ and $\bar{G}(x)$ are *lattice valued*, then the distribution of N may still be tail equivalent to $\Phi(n)^{-1}$, as in Proposition 2.3, but the constant in front of $\Phi(n)^{-1}$ may be different from $\Gamma(\alpha+1)$, e.g., if $\mathbb{P}[L > n] \sim e^{-pn}, p > 0$ and $\mathbb{P}[A > n] \sim e^{-qn}, q > 0$, then this constant is between $e^{-p}\Gamma(1 + p/q)$ and $e^{p}\Gamma(1 + p/)$.

Before moving to the proof, we state two straightforward consequences of the preceding theorems; see also Theorem 1 and Corollary 1 in [11]. The following corollary allows $\bar{F}$ and $\bar{G}$ to have exponential type distributions, and the corresponding result for T was first derived in Theorem 7 of [18].

Corollary 2.1 *Assume that $\bar{G}(x) \sim e^{-\beta x}$ and $\bar{F}(x) \sim a x^{b} e^{-\delta x}$ where $b \in \mathbb{R}$ and $a, \beta > 0$, then,*

$$\mathbb{P}[N > n] \sim a\Gamma\left(\frac{\delta}{\beta} + 1\right)\beta^{-b}\frac{(\log t)^{b}}{t^{\frac{\delta}{\beta}}}. \tag{2.6}$$

Proof: It is easy to verify that, as $x \to \infty$,

$$\bar{F}^{-1}(x) \sim a^{-1}\beta^{b}\left(\log \bar{G}^{-1}(x)\right)^{-b}\bar{G}(x)^{-\frac{\delta}{\beta}},$$

and, therefore, we can choose

$$\Phi(x) = a^{-1}\beta^{b}\left(\log x\right)^{-b}x^{\frac{\delta}{\beta}},$$

which, by using Theorem 2.5, finishes the proof. □

The following corollary allows $\bar{F}$ and $\bar{G}$ to have normal-like distributions, i.e., much lighter tails than exponential distributions, as shown in Corollary 1 of [11] (see also Corollary 2.2 in [2]).

Corollary 2.2 *Suppose $\bar{G}(x) = \mathbb{P}[|N(0, \sigma_A^2)| > x]$ and $\bar{F}(x) = \mathbb{P}[|N(0, \sigma_L^2)| > x]$, where $N(0, \sigma^2)$ is a Gaussian random variable with mean zero and variance σ^2, then,*

$$\mathbb{P}[N > n] \sim \Gamma\left(\alpha+1\right)\alpha^{-1/2}\frac{(\pi\log n)^{\frac{1}{2}(\alpha-1)}}{n^{\alpha}}, \tag{2.7}$$

where $\alpha = {\sigma_A}^2/{\sigma_L}^2$.

Proof: First, notice that

$$\mathbb{P}[|N(0,\sigma^2)| > x] \sim \frac{2\sigma}{\sqrt{2\pi}x} e^{-\frac{x^2}{2\sigma^2}},$$

and therefore, recalling $\alpha = {\sigma_A}^2/{\sigma_L}^2$, we obtain

$$\bar{F}(x) \sim \pi^{\frac{1}{2}(\alpha-1)} \alpha^{-1/2} \left(-\log \bar{G}(x)\right)^{\frac{1}{2}(\alpha-1)} \left(\bar{G}(x)\right)^{\alpha}.$$

Hence, $\bar{F}(x)$ and $\bar{G}(x)$ satisfy the assumption of Theorem 2.1 with

$$\Phi(x) = \alpha^{1/2} (\pi \log x)^{\frac{1}{2}(1-\alpha)} x^{\alpha},$$

which implies (2.7). □

Next, we present the proofs for the preceding Propositions 2.2, 2.3 and Theorem 2.1. Note that the following proof represents a basis for the other proofs in this paper.

Proof: [of Proposition 2.2] Notice that the number of retransmissions is geometrically distributed given the packet size L,

$$\mathbb{P}[N > n \mid L] = (1 - \bar{G}(L))^n$$

and, therefore,

$$\mathbb{P}[N > n] = \mathbb{E}[(1 - \bar{G}(L))^n]. \tag{2.8}$$

Since $\Phi(x)$ is eventually non-decreasing, there exists x_0 such that for all $x > x_0$, $\Phi(x)$ has an inverse function $\Phi^{\leftarrow}(x)$. The condition (2.3) implies that, for $0 < \epsilon < 1$, there exists x_ϵ, such that for $x > x_\epsilon$,

$$(1-\epsilon)\bar{F}^{-1}(x) \leqslant \Phi(\bar{G}^{-1}(x)) \leqslant (1+\epsilon)\bar{F}^{-1}(x),$$

and thus, by choosing $x_\epsilon > x_0$, we obtain

$$\Phi^{\leftarrow}\left((1-\epsilon)\bar{F}^{-1}(x)\right) \leqslant \bar{G}^{-1}(x) \leqslant \Phi^{\leftarrow}\left((1+\epsilon)\bar{F}^{-1}(x)\right). \tag{2.9}$$

First, we will prove the *upper bound*. Recalling (2.8), noting that $V \triangleq \bar{F}(L)$ is a uniform random variable on $(0,1)$ (e.g., see Proposition 2.1 in Chapter 10 of [17]) and using (2.9), we obtain, for large n,

$$\begin{aligned}
\mathbb{P}[N > n] &= \mathbb{E}[(1 - \bar{G}(L))^n] \\
&= \mathbb{E}[(1 - \bar{G}(L))^n \mathbf{1}(L > x_\epsilon)] + \mathbb{E}[\left(1 - \bar{G}(L)\right)^n \mathbf{1}(L \leqslant x_\epsilon)] \\
&\leqslant \mathbb{E}\left[e^{-\frac{n}{\Phi^{\leftarrow}((1+\epsilon)V^{-1})}}\right] + \left(1 - \bar{G}(x_\epsilon)\right)^n \\
&\leqslant \mathbb{P}\left[0 \leqslant \frac{n}{\Phi^{\leftarrow}\left((1+\epsilon)V^{-1}\right)} \leqslant 1\right] + \sum_{k=0}^{\lceil \log(\epsilon n) \rceil} e^{-e^k} \mathbb{P}\left[e^k \leqslant \frac{n}{\Phi^{\leftarrow}\left((1+\epsilon)V^{-1}\right)} \leqslant e^{k+1}\right] \\
&\quad + e^{-e^{\lceil \log(\epsilon n) \rceil + 1}} \mathbb{P}\left[\frac{n}{\Phi^{\leftarrow}\left((1+\epsilon)V^{-1}\right)} > e^{\lceil \log(\epsilon n) \rceil + 1}\right] + \left(1 - \bar{G}(x_\epsilon)\right)^n \\
&\leqslant \frac{1+\epsilon}{\Phi(n)} + \sum_{k=0}^{\lceil \log(\epsilon n) \rceil} e^{-e^k} \frac{1+\epsilon}{\Phi\left(\frac{n}{e^{k+1}}\right)} + e^{-\epsilon n} + \left(1 - \bar{G}(x_\epsilon)\right)^n.
\end{aligned} \tag{2.10}$$

The condition (2.2) implies that there exist finite n_d and d, such that for $n > n_d$,

$$\frac{\Phi(n)}{\Phi(n/e)} < d,$$

resulting in, for all k satisfying $n/e^k > n_d$,

$$\frac{\Phi(n)}{\Phi\left(\frac{n}{e^{k+1}}\right)} \leqslant d^{k+2}, \tag{2.11}$$

and therefore,

$$\Phi(n) \leqslant \Phi(n_d) d^{\log\left(\frac{n}{n_d}\right)+1},$$

which, in conjunction with (2.10), yields

$$\begin{aligned}\varlimsup_{n\to\infty} \mathbb{P}[N > n]\Phi(n) &\leqslant 1 + \epsilon + \sum_{k=0}^{\infty}(1+\epsilon)e^{-e^k} d^{k+2} \\ &\quad + \varlimsup_{n\to\infty} \left(e^{-\epsilon n} + \left(1 - \bar{G}(x_\epsilon)\right)^n\right) \Phi(n_d) d^{\log\left(\frac{n}{n_d}\right)+1} \\ &= 1 + \epsilon + \sum_{k=0}^{\infty}(1+\epsilon)e^{-e^k} d^{k+2} < \infty. \end{aligned} \tag{2.12}$$

Next, we prove the *lower bound*. Recalling (2.9) and choose $n > x_0$, we obtain

$$\begin{aligned}\mathbb{P}[N > n] &= \mathbb{E}[(1 - \bar{G}(L))^n] \\ &\geqslant \left(1 - \frac{1}{n}\right)^n \mathbb{P}\left[\bar{G}(L) \leqslant \frac{1}{n}\right] \\ &\geqslant \left(1 - \frac{1}{n}\right)^n \mathbb{P}\left[\Phi^{\leftarrow}\left((1-\epsilon)\bar{F}^{-1}(L)\right) \geqslant n\right] \\ &\geqslant \left(1 - \frac{1}{n}\right)^n \frac{1-\epsilon}{\Phi(n)}, \end{aligned}$$

implying

$$\varliminf_{n\to\infty} \mathbb{P}[N > n]\Phi(n) \geqslant \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n (1-\epsilon) = e^{-1}(1-\epsilon),$$

which, in conjunction with (2.12), proves the proposition. $\square$

Proof: [of Proposition 2.3] Recalling (2.9) and choosing n large enough such that $\{\bar{G}(L) \leqslant e/n\} \subseteq \{L > x_\epsilon\}$ with x_ϵ being the same as chosen in (2.9), we obtain

$$\begin{aligned}\mathbb{P}[N > n] &= \mathbb{E}[(1 - \bar{G}(L))^n] \\ &\geqslant \left(1 - \frac{e}{n}\right)^n \mathbb{P}\left[\bar{G}(L) \leqslant \frac{e}{n}\right] \\ &\geqslant \left(1 - \frac{e}{n}\right)^n \mathbb{P}\left[\Phi^{\leftarrow}\left((1-\epsilon)\bar{F}^{-1}(L)\right) \geqslant \frac{n}{e}\right] \\ &\geqslant \left(1 - \frac{e}{n}\right)^n \frac{1-\epsilon}{\Phi\left(\frac{n}{e}\right)}, \end{aligned}$$

implying

$$\varliminf_{n\to\infty} \mathbb{P}[N>n]\Phi(n) \geqslant \varliminf_{n\to\infty}\left(1-\frac{1}{n}\right)^n \frac{(1-\epsilon)\Phi(n)}{\Phi\left(\frac{n}{e}\right)} = \infty,$$

which completes the proof. □

Proof: [of Theorem 2.1] We begin with proving (2.4). Without loss of generality, we can assume that $\Phi(x)$ is absolutely continuous and strictly monotone since, by Proposition 1.5.8 of [4], one can always find an absolutely continuous and strictly monotone function

$$\Phi^*(x) = \alpha\int_1^x \Phi(s)s^{-1}ds,\ x\geqslant 1, \tag{2.13}$$

which satisfies

$$\bar{F}^{-1}(x) \sim \Phi\left(\bar{G}^{-1}(x)\right) \sim \Phi^*\left(\bar{G}^{-1}(x)\right).$$

First, we prove the *upper bound*. Recalling (2.8), noting that $V \triangleq \bar{F}(L)$ is a uniform random variable on $(0,1)$ and using (2.9), we obtain

$$\begin{aligned}\mathbb{P}[N>n] &= \mathbb{E}[(1-\bar{G}(L))^n] \\ &= \mathbb{E}[(1-\bar{G}(L))^n\mathbf{1}(L\geqslant x_\epsilon)] + \mathbb{E}[\left(1-\bar{G}(L)\right)^n\mathbf{1}(L<x_\epsilon)] \\ &\leqslant \mathbb{E}\left[e^{-\frac{n}{\Phi^{(-1)}((1+\epsilon)V^{-1})}}\right] + \left(1-\bar{G}(x_\epsilon)\right)^n. \end{aligned}\tag{2.14}$$

Then, by choosing integers m and n_d (as in (2.11)) and noting that $\Phi(n)$ is regularly varying, the preceding inequality yields, for large n,

$$\begin{aligned}\mathbb{P}[N>n] &\leqslant \mathbb{E}\left[e^{-\frac{n}{\Phi^{(-1)}((1+\epsilon)V^{-1})}}\mathbf{1}\left(0\leqslant \frac{n}{\Phi^{\leftarrow}((1+\epsilon)V^{-1})}\leqslant e^m\right)\right] \\ &\quad + \sum_{k=m}^{\log\left(\frac{n}{n_d}\right)-1} e^{-e^k}\mathbb{P}\left[e^k\leqslant \frac{n}{\Phi^{\leftarrow}((1+\epsilon)V^{-1})}\leqslant e^{k+1}\right] + o\left(\frac{1}{\Phi(n)}\right) \\ &\leqslant \int_0^{e^m} e^{-z}\left(\frac{\Phi'(n/z)}{\Phi^2(n/z)}\frac{(1+\epsilon)n}{z^2}\right)dz + \sum_{k=m}^{\log\left(\frac{n}{n_d}\right)-1} e^{-e^k}\frac{1+\epsilon}{\Phi\left(\frac{n}{e^{k+1}}\right)} + o\left(\frac{1}{\Phi(n)}\right),\end{aligned}$$

resulting in

$$\begin{aligned}\mathbb{P}[N>n]\Phi(n) &\leqslant \int_0^{e^m}\frac{\Phi(n)}{\Phi(n/z)}\frac{\Phi'(n/z)}{\Phi(n/z)}\frac{e^{-z}(1+\epsilon)n}{z^2}dz \\ &\quad + \sum_{k=m}^{\log\left(\frac{n}{n_d}\right)-1}(1+\epsilon)e^{-e^k}\frac{\Phi(n)}{\Phi\left(\frac{n}{e^{k+1}}\right)} + \Phi(n)\left(1-\bar{G}(x_\epsilon)\right)^n \\ &\triangleq I_1+I_2+I_3. \end{aligned}\tag{2.15}$$

Since regularly varying functions are also dominantly regularly varying, the bound in (2.11) implies

$$I_2 \leqslant \sum_{k=m}^{\log\left(\frac{n}{n_d}\right)-1}(1+\epsilon)e^{-e^k}d^{k+2} \leqslant \sum_{k=m}^{\infty}(1+\epsilon)e^{-e^k}d^{k+2} < \infty. \tag{2.16}$$

For I_1, since $\Phi(n)$ is regularly varying, by the Characterisation Theorem of regular variation (e.g., see Theorem 1.4.1 of [4]) and the uniform convergence theorem of slowly varying functions (Theorem 1.2.1 of [4]), it is easy to obtain uniformly for $0 \leqslant z \leqslant e^m$, as $n \to \infty$,

$$\frac{\Phi(n)}{\Phi(n/z)} \sim z^\alpha$$

and, recalling (2.13),

$$\frac{\Phi'(n/z)}{\Phi(n/z)} = \frac{z\alpha}{n},$$

which implies

$$I_1 \sim \int_0^{e^m} (1+\epsilon)\alpha e^{-z} z^{\alpha-1} dz. \tag{2.17}$$

Furthermore, $\Phi(n)$ being regularly varying implies that $I_3 \to 0$ as $n \to \infty$. Thus, passing $n \to \infty$ in (2.15), recalling (2.16) and then passing $m \to \infty$, $\epsilon \to 0$, we obtain

$$\mathbb{P}[N > n]\Phi(n) \lesssim \int_0^\infty \alpha e^{-z} z^{\alpha-1} dz = \Gamma(\alpha+1). \tag{2.18}$$

As for the *lower bound*, the proof follows similar arguments, and the details are presented in Subsection 4.3. The same subsection also contains the proof of the statement ii) of the theorem. □

The condition of $\Phi(\cdot)$ being dominantly varying is basically necessary in order for $\mathbb{P}[N > n] \approx \Phi(n)^{-1}$ to hold. As shown in Proposition 2.4, this tail equivalence basically does not hold if $\Phi(\cdot)$ is not dominantly varying, e.g., if $\Phi(\cdot)^{-1}$ is lognormal. Here, we further characterize the behavior of the lognormal type distributions in the following proposition.

Proposition 2.4 *If* $\log(\Phi(x)) = \lambda(\log x)^\delta$, $\delta > 1, \lambda > 0$, *then, under the condition (2.3), we obtain*

$$\lim_{n\to\infty} \frac{\log\left(\mathbb{P}[N > n]^{-1}\right) - \log(\Phi(x))}{(\log\log n)(\log n)^{\delta-1}} = -\lambda\delta(\delta-1).$$

The **proof** of this proposition is presented in Subsection 4.4.

Remark 4 In Proposition 2.4, it can be easily verified that $\Phi(\cdot)$ is not dominantly regularly varying, and therefore, according to Propositions 2.2 and 2.3, we know $\mathbb{P}[N > n]\Phi(x) \to \infty$ as $n \to \infty$. However, Proposition 2.4 further characterize how fast $\mathbb{P}[N > n]\Phi(n)$ goes to infinity in the logarithmic scale, which also implies a weaker result

$$\lim_{n\to\infty} \frac{\log\left(\mathbb{P}[N > n]^{-1}\right)}{\log(\Phi(n))} = 1.$$

In the following theorem we extend the preceding logarithmic limit under a more general condition on $\Phi(\cdot)$.

Theorem 2.2 *If an eventually non-decreasing function* $\Phi(x) \triangleq e^{l(x)}$ *satisfies (2.1) where* $l(x)$ *is slowly varying with*

$$\lim_{x\to\infty} \frac{l\left(\frac{x}{l(x)}\right)}{l(x)} = 1, \tag{2.19}$$

then,

$$\lim_{n\to\infty} \frac{\log\left(\mathbb{P}[N > n]^{-1}\right)}{\log\Phi(n)} = 1. \tag{2.20}$$

Remark 5 Note that if $\log(\Phi(x)) = e^{(\log x)^\delta}$ then the condition (2.19) is satisfied if $0 < \delta < 1/2$ and it does not hold if $\delta \geqslant 1/2$, which can be easily verified. Furthermore, if $\log(\Phi(x)) = \Psi(\log x)$ where $\Psi(x)$ is regularly varying, e.g., $\Phi(x)^{-1}$ being lognormal, then the condition (2.19) also holds, which is stated in the following corollary.

Remark 6 (Lattice variables) When L is lattice valued, it is easy to see from the proof of Theorem 2.2 that, if there exists a continuous random variable L^* such that $\log \mathbb{P}[L^* > x] \sim \log \mathbb{P}[L > x]$ as $x \to \infty$, or equivalently, if there exists a continuous negative non-increasing function $q(x)$ such that $\log \mathbb{P}[L > x] \sim q(x)$, then Theorem 2.2 still holds, e.g., when L has a geometric or Poisson distribution. To rigorously prove this claim, one can use similar arguments as in the proof of Theorem 3.1 in Section 3 of this paper. Note that this remark also applies to other logarithmic asymptotics, e.g., see Corollary 2.3, Propositions 2.5 and 2.6, and Theorems 2.3, 2.4, 2.6 and 2.7.

Corollary 2.3 *If a regularly varying function $\Psi(\cdot)$ with a non-negative index satisfies*

$$\lim_{x\to\infty} \frac{\log \bar{F}(x)^{-1}}{\Psi\left(\log \bar{G}(x)^{-1}\right)} = 1$$

and, in addition, is eventually non-decreasing when $\Psi(\cdot)$ is slowly varying, then, we have

$$\lim_{n\to\infty} \frac{\log\left(\mathbb{P}[N > n]^{-1}\right)}{\Psi(\log n)} = 1.$$

Remark 7 This result, or more precisely Theorem 2.6 in Subsection 2.2, implies parts (1:1), (2:1) and (2:2) of Theorem 2.1 in [2] and extends Theorem 2 in [11].

Proof: [of Corollary 2.3] For a regularly varying function $\Psi(\cdot)$, it is easy to verify that $l(x) = \Psi(\log(x))$ satisfies

$$\lim_{x\to\infty} \frac{l\left(\frac{x}{l(x)}\right)}{l(x)} = \lim_{x\to\infty} \frac{\Psi\left(\log x - \log \Psi(\log(x))\right)}{\Psi(\log(x))} = 1,$$

and therefore, by Theorem 2.2, we prove the corollary. $\square$

Remark 8 Note that, in conjunction with Remark 5, the condition (2.19) is close to necessary since the result (2.20) does not hold if $\log(\Phi(x)) = e^{(\log x)^\delta}$, $1/2 < \delta < 1$, as can be seen from the following proposition.

Proposition 2.5 *If* $\log(\Phi(x)) = e^{\lambda(\log x)^\delta}$, $1/2 < \delta < 1$, $\lambda > 0$, *then, under the condition (2.1), we obtain*

$$\log\left(\log\left(\mathbb{P}[N > n]^{-1}\right)\right) - \log\left(\log(\Phi(x))\right) \sim -\delta\lambda^2(\log n)^{2\delta-1}.$$

The **proof** of this proposition is presented in Subsection 4.5.

Remark 9 Note that this result implies that, for $0 < \epsilon < 1$ and n large,

$$0 \leqslant \frac{\log\left(\mathbb{P}[N > n]^{-1}\right)}{\log \Phi(n)} \leqslant e^{-(1-\epsilon)\alpha\lambda^2(\log n)^{2\alpha-1}} \to 0,$$

which contrasts the limit in (2.20).

Proof: [of Theorem 2.2] Since $\Phi(x)$ is eventually non-decreasing, there exists x_0 such that for all $x > x_0$, $\Phi(x)$ has an inverse function $\Phi^{\leftarrow}(x)$. The condition (2.1) implies that, for $0 < \epsilon < 1$, there exists x_ϵ, such that for $x > x_\epsilon$,

$$\bar{F}^{-(1-\epsilon)}(x) \leqslant \Phi(\bar{G}^{-1}(x)) \leqslant \bar{F}^{-(1+\epsilon)}(x),$$

thus, choosing $x_\epsilon > x_0$, we obtain

$$\Phi^{\leftarrow}\left(\bar{F}^{-(1-\epsilon)}(x)\right) \leqslant \bar{G}^{-1}(x) \leqslant \Phi^{\leftarrow}\left(\bar{F}^{-(1+\epsilon)}(x)\right). \tag{2.21}$$

First, we prove the *upper bound*. Recalling (2.8), noting that $V \triangleq \bar{F}(L)$ is a uniform random variable on $(0,1)$, and using (2.21), we obtain, for integer y and large n,

$$\begin{aligned}
\mathbb{P}[N > n] &= \mathbb{E}[(1-\bar{G}(L))^n] \\
&= \mathbb{E}[(1-\bar{G}(L))^n \mathbf{1}(L > x_\epsilon)] + \mathbb{E}[\left(1-\bar{G}(L)\right)^n \mathbf{1}(L \leqslant x_\epsilon)] \\
&\leqslant \mathbb{E}\left[e^{-\frac{n}{\Phi^{\leftarrow}\left(V^{-(1+\epsilon)}\right)}}\right] + \left(1-\bar{G}(x_\epsilon)\right)^n \\
&\leqslant \sum_{k=0}^{y} e^{-k}\mathbb{P}\left[k \leqslant \frac{n}{\Phi^{\leftarrow}\left(V^{-(1+\epsilon)}\right)} \leqslant k+1\right] + e^{-(y+1)} + \left(1-\bar{G}(x_\epsilon)\right)^n,
\end{aligned}$$

which, by Proposition 1.1, noting $\Phi(x) = e^{l(x)}$ and choosing $y = \lceil l(n) \rceil - 1$, implies

$$\begin{aligned}
\mathbb{P}[N > n] &\leqslant \sum_{k=0}^{\lceil l(n)\rceil - 1} e^{-k-\frac{1}{1+\epsilon} l\left(\frac{n}{k+1}\right)} + e^{-l(n)} + o\left(\mathbb{P}[N > n]\right) \\
&\leqslant \lceil l(n) \rceil e^{-\frac{1}{1+\epsilon} l\left(\frac{n}{\lceil l(n)\rceil}\right)} + e^{-l(n)} + o\left(\mathbb{P}[N > n]\right).
\end{aligned} \tag{2.22}$$

From (2.1), it is easy to see that $l(x)$ increases to infinity when $x \to \infty$ and, since $l(x)$ is slowly varying, by (2.19) and (2.22), we obtain

$$\varliminf_{n\to\infty} \frac{\log \mathbb{P}[N > n]^{-1}}{l(n)} \geqslant 1. \tag{2.23}$$

Next, we prove the *lower bound*. Recalling (2.21) and choosing n large enough, we obtain

$$\begin{aligned}
\mathbb{P}[N > n] &= \mathbb{E}[(1-\bar{G}(L))^n] \\
&\geqslant \left(1 - \frac{1}{n}\right)^n \mathbb{P}\left[\bar{G}(L) \leqslant \frac{1}{n}\right] \\
&\geqslant \left(1 - \frac{1}{n}\right)^n \mathbb{P}\left[\Phi^{\leftarrow}\left(\bar{F}^{-(1-\epsilon)}(L)\right) \geqslant n\right] \\
&\geqslant \left(1 - \frac{1}{n}\right)^n \frac{1}{\Phi(n)^{\frac{1}{1-\epsilon}}},
\end{aligned}$$

implying

$$\varlimsup_{n\to\infty} \frac{\log \mathbb{P}[N > n]^{-1}}{l(n)} \leqslant \frac{1}{1-\epsilon},$$

which, by passing $\epsilon \to 0$ and in conjunction with (2.23), proves the theorem. □

2.1.2 Medium Heavy (Weibull) Asymptotics

In the preceding subsection, we studied the scenario when the distribution of N is heavier than any Weibull distribution. Specifically, we establish the necessary conditions under which $\mathbb{P}[N > n] \approx \Phi^{-1}(n)$ holds when the separation between $\mathbb{P}[L > x]$ and $\mathbb{P}[A > x]$ can be characterized in the form of $\Phi(x) = e^{l(x)}$ with $l(x)$ being slowly varying. In this subsection, we further increase the separation in the sense that $\Phi(x) = e^{R_\beta(x)}$ with $R_\beta(x)$ being regularly varying of index $\beta > 0$, and under this condition the distribution of N is shown to be of Weibull type. In this situation, the tail equivalence developed in the preceding subsection does not hold anymore and admits a different form, as stated in the following theorem.

Theorem 2.3 *If an eventually non-decreasing function $\Phi(x) \triangleq e^{R_\beta(x)}$ satisfies (2.1) where $R_\beta(x) \equiv x^\beta l(x)$, $\beta > 0$ is regularly varying with $l(x)$ satisfying*

$$\lim_{x\to\infty} \frac{l\left(\left(\frac{x}{l(x)}\right)^{\frac{1}{1+\beta}}\right)}{l(x)} = 1, \tag{2.24}$$

then,

$$\lim_{n\to\infty} \frac{\log\left(\mathbb{P}[N > n]^{-1}\right)}{\left(\log \Phi(n)\right)^{\frac{1}{\beta+1}}} = \beta^{\frac{1}{\beta+1}} + \beta^{-\frac{\beta}{\beta+1}}. \tag{2.25}$$

Remark 10 This theorem, or more precisely Theorem 2.7 of the following Subsection 2.2, implies part (1:2) of Theorem 2.1 in [2], and provides a more precise logarithmic asymptotics instead of a double logarithmic limit that was proved in [2]. Furthermore, although the condition (2.24) appears complicated, it is easy to check that any slowly varying function $l(x) = l_1(\log x)$ satisfies it, where $l_1(\cdot)$ is also a slowly varying function.

Proof: [of Theorem 2.3] First, we begin with proving the *upper bound*. Following the same approach as in the proof of Theorem 2.2, we obtain, for $\epsilon > 0$, integer y and n large enough,

$$\begin{aligned}
\mathbb{P}[N > n] &\leqslant \sum_{k=0}^{y-1} e^{-k}\mathbb{P}\left[k \leqslant \frac{n}{\Phi^{\leftarrow}(V^{-(1+\epsilon)})} \leqslant k+1\right] + e^{-y} + o\left(\mathbb{P}[N > n]\right) \\
&\leqslant \sum_{k=0}^{y-1} e^{-k - \frac{1}{1+\epsilon} R_\beta\left(\frac{n}{k+1}\right)} + e^{-y} + o\left(\mathbb{P}[N > n]\right).
\end{aligned} \tag{2.26}$$

Using the same argument as in (2.13), we can find an absolutely continuous and strictly increasing function $R^*_\beta(u) \triangleq \beta \int_1^u R_\beta(s)s^{-1}ds, u \geqslant 1$ that is a modified version of $R_\beta(u)$. This newly constructed function $R^*_\beta(u)$ satisfies that, for $0 < \epsilon < 1$, there exists $y_\epsilon > 0$, such that $(1-\epsilon)R^*_\beta(u) < R_\beta(u) < (1+\epsilon)R^*_\beta(u)$ for $u > y_\epsilon$. Therefore, for $0 < x < n/y_\epsilon$,

$$x + \frac{1}{1+\epsilon} R_\beta\left(\frac{n}{x}\right) \geqslant x + \frac{1-\epsilon}{1+\epsilon} R^*_\beta\left(\frac{n}{x}\right),$$

and, for $u \geqslant 1$,

$$\left(R^*_\beta(u)\right)' = \beta u^{\beta-1} l(u). \tag{2.27}$$

Choosing $y = \lceil n/y_\epsilon \rceil$ in (2.26) and using the asymptotic equivalence relationship between $R_\beta(\cdot)$ and $R^*_\beta(\cdot)$, we obtain

$$\begin{aligned}\mathbb{P}[N > n] &\leqslant \sum_{k=0}^{\left\lceil \frac{n}{y_\epsilon} \right\rceil - 1} e^{-k - \frac{1}{(1+\epsilon)} R_\beta\left(\frac{n}{k+1}\right)} + e^{-\frac{n}{y_\epsilon}} + o\left(\mathbb{P}[N > n]\right) \\ &\leqslant \sum_{k=0}^{\left\lceil \frac{n}{y_\epsilon} \right\rceil - 1} e^{-k - \frac{1-\epsilon}{1+\epsilon} R^*_\beta\left(\frac{n}{k+1}\right)} + o\left(\mathbb{P}[N > n]\right). \end{aligned} \tag{2.28}$$

Next, let $f(x) = x + R^*_\beta\left(n/x\right)(1-\epsilon)/(1+\epsilon)$, and suppose that $f(x)$ reaches the maximum at x^* for $0 < x \leqslant n/y_\epsilon$. From (2.27), it is easy to check that

$$f'(x) = 1 - \frac{1-\epsilon}{1+\epsilon}\left(R^*_\beta\left(\frac{n}{x}\right)\right)' \frac{n}{x^2} = 1 - \frac{1-\epsilon}{1+\epsilon}\frac{\beta}{n}\frac{n^{\beta+1}}{x^{\beta+1}} l\left(\frac{n}{x}\right).$$

Then, define $g(u) \triangleq u^{\beta+1} l\left(u\right)$, and use the same argument as in constructing $R^*_\beta(\cdot)$, we can find an absolutely continuous and strictly increasing function $g^*(u) \triangleq \beta \int_1^u u^\beta l\left(u\right) ds, u \geqslant 1$, such that $(1-\epsilon)g(u) < g^*(u) < (1+\epsilon)g(u), u > u_\epsilon$ for $u_\epsilon > 0$. Therefore, for $0 < x < n/u_\epsilon$, we obtain,

$$1 - \frac{1}{1+\epsilon}\frac{\beta}{n} g^*\left(\frac{n}{x}\right) < f'(x) = 1 - \frac{1-\epsilon}{1+\epsilon}\frac{\beta}{n} g\left(\frac{n}{x}\right) < 1 - \frac{1-\epsilon}{(1+\epsilon)^2}\frac{\beta}{n} g^*\left(\frac{n}{x}\right),$$

where, as shown in the preceding inequalities, the lower and upper bound of $f'(x)$ are two monotonically increasing functions for $0 < x < n$.

Now, define

$$x_1 \triangleq \left(\frac{(1-\epsilon)^3}{(1+\epsilon)^2}\right)^{\frac{1}{\beta+1}} \beta^{\frac{1}{\beta+1}} n^{\frac{\beta}{\beta+1}} l\left(n\right)^{\frac{1}{\beta+1}}$$

and

$$x_2 \triangleq (1+\epsilon)^{\frac{1}{\beta+1}} \beta^{\frac{1}{\beta+1}} n^{\frac{\beta}{\beta+1}} l\left(n\right)^{\frac{1}{\beta+1}}.$$

It is easy to see that, by condition (2.24), for n large enough,

$$f'(x_1) \leqslant 1 - \frac{1-\epsilon}{(1+\epsilon)^2}\frac{\beta}{n} g^*\left(\frac{n}{x_1}\right) < 1 - \left(\frac{1-\epsilon}{1+\epsilon}\right)^2 \frac{\beta}{n} g\left(\frac{n}{x_1}\right) < 0,$$

and

$$f'(x_2) \geqslant 1 - \frac{1}{1+\epsilon}\frac{\beta}{n} g^*\left(\frac{n}{x_2}\right) > 1 - \frac{\beta}{n} g\left(\frac{n}{x_2}\right) > 0,$$

which implies that, there exist $n_\epsilon > 0$ such that for all $n > n_\epsilon$,

$$x_1 < x^* < x_2. \tag{2.29}$$

Therefore, using (2.28), (2.29) and recalling $R_\beta(u) < (1+\epsilon) R^*_\beta(u)$ yields

$$\begin{aligned}\mathbb{P}[N > n] &\leqslant \left\lceil \frac{n}{y_\epsilon} \right\rceil e^{1 - f(x^*)} + o\left(\mathbb{P}[N > n]\right) \\ &\leqslant \left\lceil \frac{n}{y_\epsilon} \right\rceil e^{1 - x_1 - \frac{1}{(1+\epsilon)^2} R_\beta\left(\frac{n}{x_2}\right)} + o\left(\mathbb{P}[N > n]\right),\end{aligned}$$

resulting in

$$\underline{\lim}_{n\to\infty} \frac{\log\left(\mathbb{P}[N>n]^{-1}\right)}{n^{\frac{\beta}{\beta+1}} l(n)^{\frac{1}{\beta+1}}} \geqslant \left(\frac{(1-\epsilon)^3}{(1+\epsilon)^2}\right)^{\frac{1}{\beta+1}} \beta^{\frac{1}{\beta+1}} + (1+\epsilon)^{-\frac{\beta}{\beta+1}-2} \beta^{-\frac{\beta}{\beta+1}}.$$

Passing $\epsilon \to 0$ in the preceding inequality yields

$$\underline{\lim}_{n\to\infty} \frac{\log\left(\mathbb{P}[N>n]^{-1}\right)}{n^{\frac{\beta}{\beta+1}} l(n)^{\frac{1}{\beta+1}}} \geqslant \beta^{\frac{1}{\beta+1}} + \beta^{-\frac{\beta}{\beta+1}}. \tag{2.30}$$

Now, we proceed with proving the *lower bound*. By recalling the condition (2.21) and using $1-x \geqslant e^{-(1+\epsilon)x}$ for x small enough, we obtain, for n large enough and $x_0 > 0$,

$$\begin{aligned}
\mathbb{P}[N>n] &\geqslant \mathbb{E}[(1-\bar{G}(L))^n \mathbf{1}(L \geqslant x_\epsilon)] \geqslant \mathbb{E}[e^{-(1+\epsilon)\bar{G}(L)n} \mathbf{1}(L \geqslant x_\epsilon)] \\
&\geqslant \mathbb{E}\left[e^{-\frac{(1+\epsilon)n}{\Phi^{\leftarrow}\left(V^{-(1-\epsilon)}\right)}} \mathbf{1}(V \leqslant \bar{F}(x_\epsilon))\right] \geqslant e^{-x_0}\mathbb{P}\left[\frac{(1+\epsilon)n}{\Phi^{\leftarrow}\left(V^{-(1-\epsilon)}\right)} \leqslant x_0, V \leqslant \bar{F}(x_\epsilon)\right] \\
&= e^{-x_0}\Phi\left(\frac{(1+\epsilon)n}{x_0}\right)^{-\frac{1}{1-\epsilon}} = e^{-x_0 - \frac{1}{1-\epsilon}R_\beta\left(\frac{(1+\epsilon)n}{x_0}\right)},
\end{aligned}$$

since $\{(1+\epsilon)n/\Phi^{\leftarrow}\left(V^{-(1-\epsilon)}\right) \leqslant x_0\}$ implies $\{V \leqslant \bar{F}(x_\epsilon)\}$ for all n large enough. Next, by choosing $x_0 = \beta^{\frac{1}{\beta+1}} n^{\frac{\beta}{\beta+1}} l\,(n)^{\frac{1}{\beta+1}}$, using the condition (2.24), and then passing $n \to \infty$ as well as $\epsilon \to 0$, yields,

$$\overline{\lim}_{n\to\infty} \frac{\log\left(\mathbb{P}[N>n]^{-1}\right)}{n^{\frac{\beta}{\beta+1}} l(n)^{\frac{1}{\beta+1}}} \leqslant \beta^{\frac{1}{\beta+1}} + \beta^{-\frac{\beta}{\beta+1}}. \tag{2.31}$$

Finally, combining (2.30) and (2.31) finishes the proof. □

2.1.3 Nearly Exponential Asymptotics

In the preceding subsection, the functional separation between $\mathbb{P}[L > x]$ and $\mathbb{P}[A > x]$ can be characterized in the form of $\Phi(x) = e^{R_\gamma(x)}$ with $R_\gamma(x)$ being regularly varying. In this subsection, we investigate the situation when the separation in terms of $\Phi(x)$ is even larger than $e^{R_\gamma(x)}$, which leads to the nearly exponential asymptotics for $\mathbb{P}[N > n]$ in the following proposition and Theorem 2.4.

Proposition 2.6 *If* $\log(\bar{F}^{-1}(x)) \sim e^{\left(\log(\bar{G}^{-1}(x))\right)^\delta}$, $\delta > 1$, *then,*

$$\log\left(\log\left(\mathbb{P}[N>n]^{-1}\right)\right) - \log n + (\log n)^{\frac{1}{\delta}} \sim \frac{1}{\delta}(\log n)^{\frac{2}{\delta}-1}.$$

Remark 11 Observe that $\delta = 2$ represents another critical point since $(\log n)^{2/\delta-1}$ converges to 0 or ∞ if $\delta > 2$ or $1 < \delta < 2$, respectively. Furthermore, the result shows that $\mathbb{P}[N > n] \approx \exp\left(-n/e^{(\log n)^{1/\delta}}\right)$, which means that N is nearly exponential because $e^{(\log n)^{1/\delta}}$ is slowly varying for $\delta > 1$ (see p. 16 in [4]). In addition, informally speaking, we point out that the case $\delta = 1$ corresponds to the Weibull case already covered by Theorem 2.3 in Subsection 2.1.2, meaning that this proposition describes the change in functional behavior on the boundary between the Weibull case and the nearly exponential one.

Proof: First, we prove the *upper bound*. Following the same approach as in the proof of Theorem 2.2, we obtain, for $\epsilon > 0$,

$$\mathbb{P}[N > n] \leqslant \sum_{k=0}^{n-1} e^{-k - \frac{1}{1+\epsilon} e^{(\log n - \log(k+1))^\delta}} + o\left(\mathbb{P}[N > n]\right). \tag{2.32}$$

Suppose that $f(x) \triangleq x + \frac{1}{1+\epsilon} e^{(\log n - \log x)^\delta}$ reaches the minimum at x^*. It is easy to see that $f'(x) = 1 - e^{(\log n - \log x)^\delta}/((1+\epsilon)x)$ is an increasing function in x on $(0, n)$. For $0 < \epsilon < 1$ define

$$x_1 \triangleq \frac{n}{e^{\left(\log n - (1-\epsilon)(\log n)^{1/\delta}\right)^{1/\delta}}},$$

and for n large enough, we obtain

$$f'(x_1) = 1 - \frac{e^{-(1-\epsilon)(\log n)^{1/\delta}}}{(1+\epsilon)} e^{\left(\log n - (1-\epsilon)(\log n)^{1/\delta}\right)^{1/\delta}} \leqslant 1 - \frac{e^{\epsilon(\log n)^{1/\delta} - (1-\epsilon^2)(\log n)^{2/\delta - 1}/\delta}}{1+\epsilon} < 0,$$

implying $f(x)' < 0$ for $x < x_1$. Therefore, the minimum point x^* satisfies

$$x^* \geqslant x_1. \tag{2.33}$$

Combining (2.32) and (2.33), we obtain, for n large,

$$\mathbb{P}[N > n] \leqslant n e^{1 - f(x^*)} + o\left(\mathbb{P}[N > n]\right) \leqslant n e^{1 - x_1} + o\left(\mathbb{P}[N > n]\right) < 2n e^{1 - x_1},$$

and therefore, for n large enough,

$$\log\left(\mathbb{P}[N > n]^{-1}\right) \geqslant \frac{n}{e^{(\log n - (1-\epsilon)(\log n)^{1/\delta})^{1/\delta}}} - \log(2n) - 1,$$

which implies

$$\log\left(\log\left(\mathbb{P}[N > n]^{-1}\right)\right) - \log n + (\log n)^{\frac{1}{\delta}} \gtrsim \frac{1}{\delta} (\log n)^{\frac{2}{\delta} - 1}. \tag{2.34}$$

Next, we prove the *lower bound*. By using the same arguments as in the proof of the lower bound for Theorem 2.3, we obtain, for n large enough,

$$\log\left(\mathbb{P}[N > n]^{-1}\right) \leqslant x_0 + \frac{1}{1-\epsilon} \log\left(\Phi\left(\frac{(1+\epsilon)n}{x_0}\right)\right) = x_0 + \frac{1}{1-\epsilon} e^{\left(\log\left(\frac{(1+\epsilon)n}{x_0}\right)\right)^\delta},$$

which, by choosing $x_0 = (1+\epsilon) n e^{-\left(\log n - (\log n)^{1/\delta}\right)^{1/\delta}}$, passing $n \to \infty$ and then $\delta \to 0$, yields

$$\log\left(\log\left(\mathbb{P}[N > n]^{-1}\right)\right) - \log n + (\log n)^{\frac{1}{\delta}} \lesssim \frac{1}{\delta} (\log n)^{\frac{2}{\delta} - 1}. \tag{2.35}$$

Finally, combining (2.34) and (2.35) finishes the proof.

Theorem 2.4 *If $\log(\bar{F}^{-1}(x)) \sim e^{R_\gamma\left(\bar{G}^{-1}(x)\right)}$, where $R_\gamma(\cdot)$ is regularly varying with index $\gamma > 0$, then,*

$$\log \mathbb{P}[N > n]^{-1} \sim \frac{n}{R_\gamma^{\leftarrow}(\log n)}, \tag{2.36}$$

where $R_\gamma^{\leftarrow}(\cdot)$ is the asymptotic inverse of $R_\gamma(\cdot)$ as defined in Theorem 1.5.12 on p. 28 of [4].

Remark 12 Note that the functional form in (2.36) is different from the one in (2.25) that describes the Weibull case. In principle, one could study the situations when $\Phi(\cdot)$ grows faster than three exponential scales, which would make the distributions of N even closer to the exponential one. However, from a practical point of view, these cases will basically be indistinguishable from the exponential distribution and, thus, we omit these derivations.

Proof: First, we prove the *upper bound.* Following the same approach as in the proof of Theorem 2.2, we obtain, for $0 < \epsilon < 1$ and $y > 0$,

$$\mathbb{P}[N > n] \leqslant \sum_{k=0}^{\lfloor n/y \rfloor - 1} e^{-k - \frac{1}{1+\epsilon} e^{R_\gamma\left(\frac{n}{k+1}\right)}} + o\left(\mathbb{P}[N > n]\right). \tag{2.37}$$

By using the same argument as in (2.13), we can choose $R_\gamma^*(x) = \gamma \int_1^x R(s) s^{-1} ds, x \geqslant 1$ and $R_\gamma^*(\cdot)$ is absolutely continuous, strictly increasing with an inverse $R_\gamma^{\leftarrow}(\cdot)$. Theorem 1.5.12 on p. 28 and Proposition 1.5.14 on p. 29 of [4] implies that $R_\gamma^{\leftarrow}(\cdot)$ is regularly varying with index $1/\gamma$ and is also the asymptotic inverse of $R_\gamma(\cdot)$. Therefore, there exists $y > 0$ such that for $0 < x < n/y$,

$$x + \frac{1}{1+\epsilon} e^{R_\gamma\left(\frac{n}{x}\right)} \geqslant x + \frac{1}{1+\epsilon} e^{(1-\epsilon) R_\gamma^*\left(\frac{n}{x}\right)}.$$

Suppose that $f(x) \triangleq x + e^{(1-\epsilon)R_\gamma^*\left(\frac{n}{x}\right)}/(1+\epsilon)$ reaches the minimum at x^*, and note that

$$f'(x) = 1 - \frac{1-\epsilon}{1+\epsilon} e^{(1-\epsilon)R_\gamma^*\left(\frac{n}{x}\right)} \left(R_\gamma^*\left(\frac{n}{x}\right)\right)' \frac{n}{x^2} = 1 - \frac{1-\epsilon}{1+\epsilon} e^{(1-\epsilon)R_\gamma^*\left(\frac{n}{x}\right)} \frac{\gamma R_\gamma^*\left(\frac{n}{x}\right)}{x}$$

is an increasing function for x on $(0, n/y)$. Now, defining

$$x_1 \triangleq \frac{n}{R_\gamma^{\leftarrow}\left(\frac{1}{1-\epsilon}\left(\log n - (1-\epsilon)\left(1 + \frac{1}{\gamma}\right)\log\log n\right)\right)}, \tag{2.38}$$

it is easy to check that, for all n large enough, $f'(x_1)$ is equal to

$$1 - \frac{\gamma R_\gamma^{\leftarrow}\left(\frac{1}{1-\epsilon}\left(\log n - (1-\epsilon)\left(1+\frac{1}{\gamma}\right)\log\log n\right)\right)\left(\log n - (1-\epsilon)\left(1+\frac{1}{\gamma}\right)\log\log n\right)}{(1+\epsilon)(\log n)^{(1-\epsilon)(1+\frac{1}{\gamma})}} < 0,$$

which implies that $f(x)' < 0$ for $0 < x < x_1$ and n large. Thus, the minimum point x^* satisfies

$$x^* > x_1. \tag{2.39}$$

Combining (2.37) and (2.39) yields, for n large enough,

$$\mathbb{P}[N > n] \leqslant \frac{n}{y} e^{1 - f(x^*)} + o\left(\mathbb{P}[N > n]\right) \leqslant \frac{2n}{y} e^{1 - x_1},$$

resulting in

$$\log\left(\mathbb{P}[N > n]^{-1}\right) \geqslant \frac{n}{R_\gamma^{\leftarrow}\left(\frac{1}{1-\epsilon}\left(\log n - (1-\epsilon)\left(1+\frac{1}{\gamma}\right)\log\log n\right)\right)} - \log\left(\frac{2n}{y}\right) - 1.$$

Therefore, passing $n \to \infty$ and then $\epsilon \to 0$ in the preceding inequality yields

$$\log \mathbb{P}[N > n]^{-1} \gtrsim \frac{n}{R_\gamma^{\leftarrow}(\log n)}. \tag{2.40}$$

Next, we prove the *lower bound*. By using the same arguments as in the proof of the lower bound for Theorem 2.3, we obtain, for n large enough,

$$\log\left(\mathbb{P}[N>n]^{-1}\right) \leqslant x_0 + \frac{1}{1-\epsilon}\log\left(\Phi\left(\frac{(1+\epsilon)n}{x_0}\right)\right) = x_0 + \frac{1}{1-\epsilon}e^{R_\gamma\left(\frac{(1+\epsilon)n}{x_0}\right)},$$

which, by choosing

$$x_0 = \frac{(1+\epsilon)n}{R_\gamma^{\leftarrow}\left((1-\epsilon)\log n - \frac{1}{\gamma}\log\log n\right)},$$

and noting that $R_\gamma\left(R_\gamma^{\leftarrow}(x)\right) \leqslant x/(1-\epsilon)$ for all x large enough, yields, for n large,

$$\log\left(\mathbb{P}[N>n]^{-1}\right) \leqslant x_0 + \frac{n}{(1-\epsilon)(\log n)^{\frac{1}{(1-\epsilon)\gamma}}}.$$

The preceding inequality implies

$$\log\left(\mathbb{P}[N>n]^{-1}\right) \lesssim \frac{n}{R_\gamma^{\leftarrow}(\log n)}. \tag{2.41}$$

Finally, combining (2.40) and (2.41) finishes the proof. □

2.2 Asymptotics of the Total Transmission Time T

In this subsection, we compute the asymptotics of the total transmission time T based on the previous results on $\mathbb{P}[N>n]$. Our proving technique involves the relationship between N and T described in (1.1) and the classical large deviation results. Theorem 2.5 and Theorem 2.6 characterize the exact asymptotics and logarithmic asymptotics for the very heavy case, respectively, and Theorem 2.7 derives the result for the moderate heavy (Weibull) case. Interestingly, we want to point out that, unlike Theorems 2.5 and 2.6 requiring no conditions on A (Theorem 2.5 needs $\mathbb{E}[A]<\infty$), the minimum conditions needed for Theorem 2.7, as shown by Proposition 2.7, basically involve a balance between the tail decays of $\mathbb{P}[A>x]$ and $\mathbb{P}[L>x]$.

Similarly, the corresponding results on T can be derived for the other statements on N, e.g., Propositions 2.2, 2.3, 2.4, 2.5, and Theorem 2.4. But, to avoid lengthy expositions and repetitions, we omit this derivations. In the following, let $\vee \equiv \max$.

Theorem 2.5 *If $\mathbb{E}\left[U^{(\alpha\vee 1)+\theta}\right]<\infty$, $\mathbb{E}\left[A^{1+\theta}\right]<\infty$ and $\mathbb{E}\left[L^{\alpha+\theta}\right]<\infty$ for some $\theta>0$, then, under the same conditions as in Theorem 2.1 i), i.e., $\bar{F}^{-1}(x)\sim\Phi\left(\bar{G}^{-1}(x)\right)$ with $\Phi(x)$ being regularly varying of index $\alpha>0$, we obtain, as $t\to\infty$,*

$$\mathbb{P}[T>t] \sim \frac{\Gamma(\alpha+1)(\mathbb{E}[U+A])^{\alpha}}{\Phi(t)}. \tag{2.42}$$

Remark 13 Note that $\mathbb{E}\left[L^{\alpha+\theta}\right]<\infty$ is basically a minimum condition for $\alpha>1$ since $\mathbb{E}\left[L^{\alpha-\theta}\right]=\infty$ implies $\mathbb{E}\left[T^{\alpha-\theta}\right]=\infty$ because of $T\geqslant L$, which would contradict (2.42).

The **proof** is presented in Subsection 4.6.

Theorem 2.6 *Under the same conditions of Theorem 2.2, i.e., the eventually non-decreasing function $\Phi(x)\triangleq e^{l(x)}$ satisfies (2.1) where $l(x)$ is slowly varying with*

$$\lim_{x\to\infty}\frac{l\left(\frac{x}{l(x)}\right)}{l(x)} = 1, \tag{2.43}$$

and in addition, if $\mathbb{P}[L > x] = O\left(\Phi(x)^{-(\delta+1)}\right)$ *and* $\mathbb{P}[U > x] = O\left(\Phi(x)^{-(\delta+1)}\right)$, $\delta > 0$, *then, we obtain*

$$\lim_{t\to\infty} \frac{\log\left(\mathbb{P}[T > t]^{-1}\right)}{\log(\Phi(t))} = 1. \tag{2.44}$$

Remark 14 This result implies parts (1:1), (2:1) and (2:2) of Theorem 2.1 in [2] and extends Theorem 2 in [11]. Furthermore, it shows that, if $\log \mathbb{P}[L > x]^{-1} \approx \alpha \log \mathbb{P}[A > x]^{-1}$, meaning that the hazard functions of L and A are asymptotically linear, the distribution tails of the number of transmissions and total transmission time are essentially power laws. Thus, the system can exhibit high variations and possible instability, e.g., when $0 < \alpha < 2$, the transmission time has an infinite variance and, when $0 < \alpha < 1$, it does not even have a finite mean.

Remark 15 It is easy to understand that, if the data sizes (e.g., files, packets) follow heavy-tailed distributions, the total transmission time is also heavy-tailed. However, from these two theorems, we see that even if the distributions of the data and channel characteristics are highly concentrated, e.g., when they are asymptotically proportional on the logarithmic scale (see Corollary 2.2 in Subsection 2.1.1), the heavy-tailed transmission delays can still arise.

The **proof** is presented in Subsection 4.7.

Theorem 2.7 *Under the same conditions of Theorem 2.3, i.e., the eventually non-decreasing function* $\Phi(x) \triangleq e^{R_\beta(x)}$ *satisfies (2.1) where* $R_\beta(x) = x^\beta l(x)$, $\beta > 0$ *is regularly varying with* $l(x)$ *satisfying*

$$\lim_{x\to\infty} \frac{l\left(\left(\frac{x}{l(x)}\right)^{\frac{1}{1+\beta}}\right)}{l(x)} = 1, \tag{2.45}$$

and in addition, if $\mathbb{E}[A] < \infty$, $\mathbb{P}[U > x] = O\left(e^{-(\log\Phi(x))^{(1+\delta)/(\beta+1)}}\right)$, $\delta > 0$, *and* $\mathbb{P}[L > x] = O\left(e^{-x^\xi}\right)$, $\mathbb{P}[A > x] = O\left(e^{-x^\zeta}\right)$ *with* $\xi > \beta/(\beta+1), \zeta \geqslant 0$ *satisfying* $(1-\zeta)\beta < \xi$, *then, we obtain*

$$\lim_{n\to\infty} \frac{\log\left(\mathbb{P}[T > t]^{-1}\right)}{(\log\Phi(t))^{\frac{1}{\beta+1}}} = \frac{\beta^{\frac{1}{\beta+1}} + \beta^{-\frac{\beta}{\beta+1}}}{(\mathbb{E}[A+U])^{\frac{\beta}{\beta+1}}}. \tag{2.46}$$

Remark 16 This theorem implies part (1:2) of Theorem 2.1 in [2], and provides a more precise logarithmic asymptotics instead of a double logarithmic limit. Furthermore, it is easy to check that the condition $(1-\zeta)\beta < \xi$ holds in two special cases: (i) if $\zeta \geqslant \beta/(\beta+1)$ and $\xi > \beta/(\beta+1)$ or (ii) if $\xi > \beta$ and $\zeta = 0$ (assuming no conditions for $\mathbb{P}[A > x]$ beyond $\mathbb{E}[A] < \infty$).

The **proof** is presented in Subsection 4.8. Basically, the condition $(1-\zeta)\beta < \xi$ (or equivalently $\xi/(\xi+1-\zeta) > \beta/(\beta+1)$) is needed since the following proposition shows that $\mathbb{P}[T > t]$ could have a heavier tail than predicted by (2.46) if $(1-\zeta)\beta > \xi$.

Proposition 2.7 *If* $\mathbb{P}[L > x] = e^{-x^\xi}$ *and* $\mathbb{P}[A > x] = e^{-x^\zeta}$ *with* $0 < \xi, \zeta < 1$, *then, as* $t \to \infty$,

$$\mathbb{P}[T > t] \gtrsim e^{-2t^{\xi/(\xi+1-\zeta)}}.$$

Proof: It is easy to see that, for $\delta, y > 0$,

$$\begin{aligned}
\mathbb{P}\left[T > t\right] &\geqslant \mathbb{P}\left[T > t, y < A_i < (1+\delta)y, 1 \leqslant i \leqslant \frac{t}{y}, L > (1+\delta)y\right] \\
&\geqslant \left(\mathbb{P}[y < A < (1+\delta)y]\right)^{\frac{t}{y}} \mathbb{P}[L > (1+\delta)y],
\end{aligned}$$

which, by noting that $\mathbb{P}[A > x] = e^{-x^\zeta}$ with $\zeta > 0$, yields

$$\mathbb{P}[T > t] \gtrsim (\mathbb{P}[A > y])^{\frac{t}{y}} \mathbb{P}[L > (1+\delta)y] = e^{-\left(\frac{t}{y}y^\zeta + y^\xi\right)}. \tag{2.47}$$

Choosing $y = t^{1/(\xi+1-\zeta)}$ finishes the proof. □

3 Engineering Implications

As already stated in the introduction, retransmissions are the integral component of many modern networking protocols on all communication layers from the physical to the application one. In our recent work [10, 11, 9, 12], we have shown that these protocols may result in heavy-tailed (e.g., power law) delays even if all the system components are light-tailed (superexponential). More specifically, from an engineering perspective, our main discovery is the matching between the statistical characteristics of the channel and transmitted data (e.g., packets). Basically, one can expect good or bad delay performance measured by the existence of α-moments for N and T if $\alpha \log \mathbb{P}[A > x] > \log \mathbb{P}[L > x]$ or $\alpha \log \mathbb{P}[A > x] < \log \mathbb{P}[L > x]$, respectively. Note that, if $\alpha < 1$, then the system could experience zero throughput.

On the network application layer, most of us have experienced the connection failures while downloading a large file from the Internet. This issue has been already recognized in practice where software for downloading sizable documents was developed that would save the intermediate data (checkpoints) and resume the download from the point when the connection was broken. However, our results emphasize that, in the presence of frequently failing connections, the long delays may arise even when downloading relatively small documents. Hence, we argue that one may need to adopt the application layer software for the wireless environment by introducing checkpoints even for small to moderate size documents.

Furthermore, on the physical layer, it is well known that wireless links, especially for low-powered sensor networks, have higher error rates than the wired counterparts. This may result in large delays on the data link layer due to the (IP) packet variability and channel failures. Therefore, our results suggest that packet fragmentation techniques need to be applied with special care since: if the packets are too small, they will mostly contain the packet header, which can limit the useful throughput; if the packets are too large, power law delays can deteriorate the quality of transmission. When the codewords, the basic units of packets in the physical layer, are much smaller than the maximum size of the packets, our results show that the number of retransmissions could be power law, which challenges the traditional model that assumes a geometric number of retransmissions. We believe that short codewords are realistic assumption for sensor networks, where complicated coding schemes are unlikely since the nodes have very limited computational power. In reality, packet sizes may have an upper limit (e.g., WaveLAN's maximum transfer unit is 1500 bytes), this situation may result in truncated power law distributions for T and N in the main body with a stretched (exponentiated) support in relation to the support of L (see Example 3 in Section IV of [11]) and, thus, may result in very long, although, exponentially bounded delays. The impact of truncated heavy-tailed distributions on queueing behavior was quantified in [8].

On the medium access control layer, ALOHA is a widely used protocol that provides a contention management scheme for multiple users sharing the same medium. Once a user detects a collision, it will back off for a random (exponential) period of time before trying to retransmit the collided packet. Due to its simplicity and distributed nature, ALOHA is the basis of many other protocols, such as CSMA/CD. We discovered a new phenomenon in [9] that a basic finite population ALOHA model with variable size (exponential) packets is

characterized by power law transmission delays, possibly even resulting in zero throughput; see Theorem 1 and Example 1 in [9] that characterizes and illustrates the observation respectively. This power law effect might be diminished, or perhaps eliminated, by reducing the variability of packets. However, we also show in [9] that even a slotted (synchronized) ALOHA with packets of constant size can exhibit power law delays when the number of active users is random; see Theorem 2 and Example 2 in [9]. The ALOHA system is a generalization of our study in this paper, since, informally, it can be viewed as the state dependent version of the model considered here where the distributions of L and A depend on the state of the system.

On the transport layer, most of the network protocols (e.g., TCP) use end-to-end acknowledgements for packets as an error control strategy. Namely, once the packet sent from the sender to the receiver is lost due to, e.g., finite buffers or link failures, this packet will be retransmitted by the sender. Furthermore, the number of hops that a packet traverses on its path to the destination is random, e.g., an end-user that is surfing the Web might download documents from diverse web sites. Our recent work in [10] shows that this acknowledgement mechanism, jointly with the random number of hops, may result in heavy-tailed (e.g., power law) delays. To illustrate this phenomenon, we consider the following basic example. Assume that a single data unit (packet) needs to traverse a random geometric number L of hops before reaching the destination, $\mathbb{P}[L > n] = e^{-pn}, p > 0$. Next, assume that in each hop the packet can independently (independent of L as well) be lost with probability $1 - e^{-q}, q > 0$. When a packet is lost, it is retransmitted by the sender and this procedure continues until the packet reaches its destination. Then, it is easy to see that, in conjunction with Remark 3 after Theorem 2.1, the number N of retransmissions that the sender needs to perform satisfies

$$\frac{e^{-p}\Gamma(1 + p/q)}{n^{p/q}} \leqslant \mathbb{P}[N > n] \leqslant \frac{e^{p}\Gamma(1 + p/q)}{n^{p/q}}.$$

Similarly, assuming that in each hop a packet is processed for one unit of time, we can derive that the distribution of the total transmission time T satisfies $\log(\mathbb{P}[T > t]) \sim -(p/q) \log t$.

Furthermore, when the cause of losses is due to the finiteness of buffers, i.e., a packet is lost when it sees a full buffer upon its arrival, the preceding general setup can be more precisely modeled as a sequence of random number L of tandem queues [10]. More specifically, we consider L tandem $\cdot/M/1/b$ queues with each queue being able to accommodate up to (finitely many) b packets; M stands for exponential (memoryless) service times. This model can be shown to result in heavy-tailed delays under quite general assumptions on cross traffic, network topology and routing scenarios. However, for simplicity we only present the following example. As depicted in Figure 2, suppose that the single packet, as well as the cross traffic, is sent sequentially through a chain of finite buffer queues with capacity b. Also, we assume that the cross traffic flows are i.i.d Poisson processes and the service requirements needed for processing different packets and the same packet at different queues are i.i.d. exponential random variables. Furthermore, the sender tries to transmit a single packet through the sequence of queues, and if the packet is lost, the packet will be immediately retransmitted by the sender. Then, regardless of how many hops the cross traffic flows traverse before leaving the system, the distribution of the number of retransmissions N satisfies the following Theorem 3.1.

Theorem 3.1 *If the limit $p \triangleq \lim_{n\to\infty} \log\left(\mathbb{P}[L > n]\right)/n < 0$ exists, then, there exists $0 < \alpha_1 \leqslant \alpha_2 < \infty$, such that*

$$-\alpha_2 \leqslant \varliminf_{n\to\infty} \frac{\log \mathbb{P}[N > n]}{\log n} \leqslant \varlimsup_{n\to\infty} \frac{\log \mathbb{P}[N > n]}{\log n} \leqslant -\alpha_1.$$

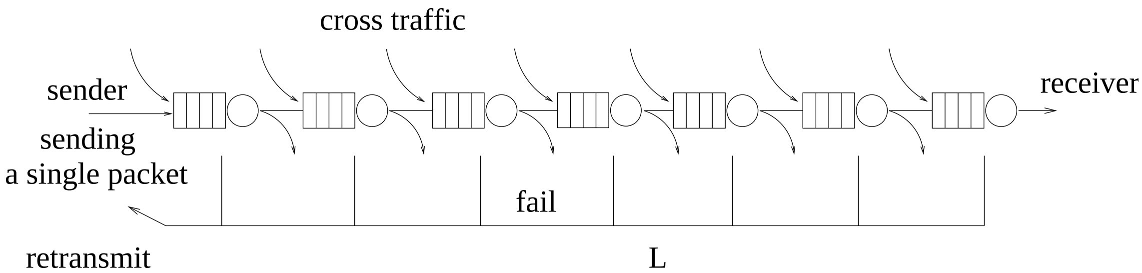

Figure 2: Tandem $\cdot/M/1/b$ queues with finite buffers

Remark 17 Note that the same result can be easily derived for T. Furthermore, due to the generality of our argument, the proof of this theorem can be applied to much more complicated routing schemes, network topologies and cross traffic conditions, e.g., with routing loops. However, such generalizations, except for complex notation, do not bring new insights, and therefore, we only study the current simple example. Further study of this model will be available in [12].

Proof: Number the sequence of queues from 1 to L sequentially. Recall that the packet is lost when it sees a full buffer upon its arrival. In order to prove the theorem, we construct two systems with independent loss probabilities at different queues that provide upper and lower bounds on the loss probabilities for the considered packet.

First, we prove the *lower bound*. Construct a system that empties queue $i+1$ whenever the considered packet begins receiving service in queue i ($1 \leqslant i < L$). Denote by C_i the event that this packet is lost when arriving at queue $i+1$. From the procedure of this construction and the memoryless property, it is easy to see that $\{C_i\}_{1\leqslant i \leqslant L}$ are i.i.d. conditional on L. Thus, we obtain, for $n_0 > 0$,

$$\mathbb{P}[N > n] \geqslant \mathbb{E}\left[\left(1 - \prod_{i=1}^{L}(1 - \mathbb{P}[C_i])\right)^n \middle| L\right] \geqslant \mathbb{E}\left[\left(1 - (1 - \mathbb{P}[C_1])^L\right)^n \mathbf{1}(L > n_0)\right]. \tag{3.1}$$

Then, we construct a continuous random variable L^* with $\mathbb{P}[L^* > x] = e^{-2px}, x \geqslant 0$, and choose n_0 large enough such that $\mathbb{P}[L^* > x] \leqslant \mathbb{P}[L > x]$ for all $x > n_0$. Therefore, by using stochastic dominance and replacing L with L^* in (3.1), we obtain

$$\mathbb{P}[N > n] \geqslant \mathbb{E}\left[\left(1 - (1 - \mathbb{P}[C_1])^{L^*}\right)^n\right] - \left(1 - (1 - \mathbb{P}[C_1])^{n_0}\right)^n, \tag{3.2}$$

which, by setting $\bar{G}(x) = (1 - \mathbb{P}[C_1])^x$, $\bar{F}(x) = \mathbb{P}[L^* > x]$ and applying Theorem 2.2, yields, for some $\alpha_2 > 0$,

$$\varliminf_{n\to\infty} \frac{\log \mathbb{P}[N > n]}{\log n} \geqslant -\alpha_2. \tag{3.3}$$

Note that this line of argument can be used to rigorously prove Remark 6.

Next, we prove the *upper bound*. Construct a system that empties queue $i+1$ whenever the considered packet begins receiving service in queue i ($1 \leqslant i < L$). Then, using this construction and similar arguments as in the proof of the lower bound, we can easily prove that there exists $\alpha_1 > 0$ such that

$$\varlimsup_{n\to\infty} \frac{\log \mathbb{P}[N > n]}{\log n} \leqslant -\alpha_1,$$

which, in conjunction with (3.3), finishes the proof. □

Finally, we would like to point out that, in addition to the preceding applications in communication networks [10, 11, 9, 12] and job processing on machines with failures [5, 18], the model studied in this paper may represent a basis for understanding more complex failure prone systems, e.g., see the recent study on parallel computing in [1].

In conclusion, we would like to emphasize that, in practice, our results provide an easily computable benchmark for measuring the tradeoff between the data statistics and channel characteristics that permits/prevents satisfactory transmission.

4 Proofs

4.1 Proof of Proposition 1.1

As stated earlier in Subsection 1.1, the proof of this proposition was originally presented in Lemma 1 of [11] and, we repeat it here for reasons of completeness.

Proof: Note that for any $\delta > 0$, there exists $t_\delta > 0$ such that, for all $0 < t < t_\delta$,

$$1 - t \geqslant e^{-\delta} e^{-t}.$$

Therefore, we can choose x_δ large enough, such that $1 - \bar{G}(x) \geqslant e^{-\delta} e^{-\bar{G}(x)}$ for all $x > x_\delta$. Then,

$$\begin{aligned} e^{\epsilon n}\mathbb{P}[N > n] &\geqslant e^{\epsilon n}\mathbb{E}\left[\left(1 - \bar{G}(L)\right)^n \mathbf{1}(L \geqslant x_\delta)\right] \geqslant e^{\epsilon n}\mathbb{E}\left[e^{n\delta} e^{-n\bar{G}(L)} \mathbf{1}(L \geqslant x_\delta)\right] \\ &\geqslant \left(e^{\epsilon - \bar{G}(x_\delta) - \delta}\right)^n \bar{F}(x_\delta). \end{aligned}$$

Thus, by selecting δ small enough and x_δ large enough, we can always make $e^{\epsilon - \bar{G}(x_\delta) - \delta} > 1$, and, by passing $n \to \infty$, we complete the proof of (1.2).

Next, we prove the corresponding result for T. Suppose that $\bar{G}(x_0) > 0$ for some $x_0 > 0$; otherwise, T will be infinite, which yields (1.3) immediately. We can always find $x_1 > x_0 > 0$, such that i.i.d. random variables $X_i \triangleq x_0 \mathbf{1}(x_0 < A_i < x_1)$ satisfy $0 < \mathbb{E}X_1 < \infty$. Now, for any $\zeta > 0$,

$$\begin{aligned} \mathbb{P}[T > t] &= \mathbb{P}\left[\sum_{i=1}^{N-1}(U_i + A_i) + L > t\right] \geqslant \mathbb{P}\left[\sum_{i=1}^{N-1} A_i \mathbf{1}(x_0 < A_i < x_1) > t\right] \\ &\geqslant \mathbb{P}\left[\sum_{i=1}^{N-1} X_i > t\right] \geqslant \mathbb{P}\left[\sum_{i=1}^{N-1} X_i > t, N \geqslant \frac{t(1+\zeta)}{\mathbb{E}X_1}\right] \\ &\geqslant \mathbb{P}\left[N > \frac{t(1+\zeta)}{\mathbb{E}X_1} + 1\right] - \mathbb{P}\left[\sum_{i=1}^{N-1} X_i \leqslant t, N > \frac{t(1+\zeta)}{\mathbb{E}X_1} + 1\right] \\ &\triangleq I_1 - I_2. \end{aligned} \tag{4.1}$$

Since, for $\bar{X}_i \triangleq \mathbb{E}[X_i] - X_i$,

$$I_2 \leqslant \mathbb{P}\left[\sum_{i \leqslant t(1+\zeta)/\mathbb{E}X_1} X_i \leqslant t\right] = \mathbb{P}\left[\sum_{i \leqslant t(1+\zeta)/\mathbb{E}X_1} \bar{X}_i \geqslant \zeta t\right], \tag{4.2}$$

it is well known (e.g., see Example 1.15 of [19]) that there exists $\eta > 0$, such that

$$I_2 \leqslant e^{-\eta t}. \tag{4.3}$$

Therefore, by (1.2), (4.1) and (4.3), we obtain that for all $0 < \epsilon < \eta$,

$$e^{\epsilon t}\mathbb{P}[T > t] \to \infty \ \text{ as } t \to \infty,$$

implying that (1.3) holds for any $\epsilon > 0$. □

4.2 Proof of Proposition 2.1

Proof: If $\log(\Phi(x))$ is slowly varying, then, for any $0 < \delta < \epsilon$, there exists $x_\delta > 0$ such that $\log(\Phi(x)) < x^\delta$ for all $x > x_\delta$. By using the condition (2.1), or equivalently (2.21), we obtain, for n large enough,

$$\begin{aligned}
\mathbb{P}[N > n] &= \mathbb{E}[(1 - \bar{G}(L))^n] \\
&\geqslant \left(1 - \frac{1}{n}\right)^n \mathbb{P}\left[\bar{G}(L) \leqslant \frac{1}{n}\right] \\
&\geqslant \left(1 - \frac{1}{n}\right)^n \mathbb{P}\left[\Phi^{\leftarrow}\left(\bar{F}^{-(1+\epsilon)}(L)\right) \geqslant n\right] \\
&\geqslant \left(1 - \frac{1}{n}\right)^n e^{-x^\delta},
\end{aligned}$$

where we use the fact that for x_ϵ chosen in (2.21) one can always select n large enough such that $\{\bar{G}(L) \leqslant 1/n\} \subset \{L > x_\epsilon\}$. Therefore, we obtain,

$$0 \leqslant \overline{\lim_{n\to\infty}} \frac{-\log \mathbb{P}[N > n]}{n^\epsilon} \leqslant \lim_{n\to\infty} \frac{-1 + n^\delta}{n^\epsilon} = 0,$$

which proves the proposition. □

4.3 Continuation of the proof of Theorem 2.1

Proof: Now, we prove the *lower bound*. For $K > 0$ and x_ϵ selected in (2.9), choosing $x_n > x_\epsilon$ with $\Phi^{\leftarrow}\left((1-\epsilon)\bar{F}(x_n)\right) = n/K$, we obtain, for large n,

$$\begin{aligned}
\mathbb{P}[N > n] &= \mathbb{E}\left[\left(1 - \bar{G}(L)\right)^n\right] \\
&\geqslant \mathbb{E}\left[\left(1 - \bar{G}(L)\right)^n \mathbf{1}(L > x_n)\right] \\
&\geqslant \mathbb{E}\left[\left(1 - \frac{1}{\Phi^{\leftarrow}\left((1-\epsilon)V^{-1}\right)}\right)^n \mathbf{1}(V < \bar{F}(x_n))\right],
\end{aligned}$$

which, by letting $z = n/\Phi^{\leftarrow}\left((1-\epsilon)V^{-1}\right)$, yields

$$\mathbb{P}[N > n]\Phi(n) \geqslant \int_0^K \left(1 - \frac{z}{n}\right)^n \frac{\Phi(n)}{\Phi(n/z)} \frac{\Phi'(n/z)}{\Phi(n/z)} \frac{(1-\epsilon)n}{z^2} dz. \tag{4.4}$$

From (4.4), by using the same approach as in deriving (2.17), we obtain, as $n \to \infty$,

$$\mathbb{P}[N > n]\Phi(n) \sim \int_0^K (1-\epsilon)\alpha e^{-z} z^{\alpha-1} dz,$$

which, by passing $K \to \infty$ and $\epsilon \to 0$, yields

$$\mathbb{P}[N > n]\Phi(n) \gtrsim \int_0^\infty \alpha e^{-z} z^{\alpha-1} dz = \Gamma(\alpha + 1). \tag{4.5}$$

Combining (2.18) and (4.5) completes the proof of (2.4).

Then, we proceed with proving (2.5). First, we prove the *lower bound*. Since $\Phi(x)$ is eventually non-decreasing, we obtain the inequality presented in (2.9) again, and therefore, for n large enough and $\epsilon > 0$,

$$\begin{aligned}
\mathbb{P}[N > n] &= \mathbb{E}[(1 - \bar{G}(L))^n] \\
&\geqslant \left(1 - \frac{\epsilon}{n}\right)^n \mathbb{P}\left[\bar{G}(L) \leqslant \frac{\epsilon}{n}\right] \\
&\geqslant \left(1 - \frac{\epsilon}{n}\right)^n \mathbb{P}\left[\Phi^{\leftarrow}\left((1-\epsilon)\bar{F}^{-1}(L)\right) \geqslant \frac{n}{\epsilon}\right] \\
&\geqslant \left(1 - \frac{\epsilon}{n}\right)^n \frac{1-\epsilon}{\Phi\left(\frac{n}{e}\right)},
\end{aligned}$$

implying

$$\varliminf_{n\to\infty} \mathbb{P}[N > n]\Phi(n) \geqslant \varliminf_{n\to\infty} \left(1 - \frac{\epsilon}{n}\right)^n \frac{(1-\epsilon)\Phi(n)}{\Phi\left(\frac{n}{\epsilon}\right)},$$

which, by passing $\epsilon \to 0$, yields

$$\varliminf_{n\to\infty} \mathbb{P}[N > n]\Phi(n) \geqslant 1. \tag{4.6}$$

Next, we prove the *upper bound*. Using a similar approach that derived (2.10), we obtain

$$\begin{aligned}
\mathbb{P}[N > n] &\leqslant \mathbb{E}\left[e^{-\frac{n}{\Phi^{\leftarrow}((1+\epsilon)V^{-1})}}\right] + \left(1 - \bar{G}(x_\epsilon)\right)^n \\
&\leqslant \mathbb{P}\left[0 \leqslant \frac{n}{\Phi^{\leftarrow}\left((1+\epsilon)V^{-1}\right)} \leqslant e^m\right] \\
&\quad + \sum_{k=m}^{\lceil \log(\epsilon n)\rceil} e^{-e^k} \mathbb{P}\left[e^k \leqslant \frac{n}{\Phi^{\leftarrow}\left((1+\epsilon)V^{-1}\right)} \leqslant e^{k+1}\right] + o\left(\frac{1}{\Phi(n)}\right) \\
&\leqslant \frac{1+\epsilon}{\Phi\left(\frac{n}{e^m}\right)} + \sum_{k=m}^{\lceil \log(\epsilon n)\rceil} e^{-e^k} \frac{1+\epsilon}{\Phi\left(\frac{n}{e^{k+1}}\right)} + o\left(\frac{1}{\Phi(n)}\right),
\end{aligned}$$

resulting in

$$\mathbb{P}[N > n]\Phi(n) \leqslant \frac{(1+\epsilon)\Phi(n)}{\Phi\left(\frac{n}{e^m}\right)} + \sum_{k=m}^{\lceil \log(\epsilon n)\rceil} e^{-e^k} \frac{(1+\epsilon)\Phi(n)}{\Phi\left(\frac{n}{e^{k+1}}\right)} + o\left(1\right). \tag{4.7}$$

Note that the second term in the right hand side of (4.7) is always finite because of (2.11) and, by passing $n \to \infty$ and then $m \to \infty$ in (4.7), we obtain

$$\varlimsup_{n\to\infty} \mathbb{P}[N > n]\Phi(n) \leqslant 1. \tag{4.8}$$

Combining (4.6) and (4.8) finishes the proof of (2.5). $\square$

4.4 Proof of Proposition 2.4

Proof: First, we prove the *lower bound*. By recalling the condition (2.3), or equivalently (2.9), and using $1-x \geqslant e^{-(1+\epsilon)x}$ for x small enough, we obtain, for n large enough and $x_0 > 0$,

$$\begin{aligned}\mathbb{P}[N>n] &\geqslant \mathbb{E}[(1-\bar{G}(L))^n \mathbf{1}(L \geqslant x_\epsilon)] \geqslant \mathbb{E}[e^{-(1+\epsilon)\bar{G}(L)n}\mathbf{1}(L \geqslant x_\epsilon)] \\ &\geqslant \mathbb{E}\left[e^{-\frac{(1+\epsilon)n}{\Phi^{\leftarrow}((1-\epsilon)V^{-1})}}\mathbf{1}(V \leqslant \bar{F}(x_\epsilon))\right] \geqslant e^{-x_0}\mathbb{P}\left[\frac{(1+\epsilon)n}{\Phi^{\leftarrow}\left((1-\epsilon)V^{-1}\right)} \leqslant x_0, V \leqslant \bar{F}(x_\epsilon)\right] \\ &= e^{-x_0}(1-\epsilon)\Phi\left(\frac{(1+\epsilon)n}{x_0}\right)^{-1} = (1-\epsilon)e^{-x_0-\lambda\left(\log n - \log\left(\frac{x_0}{1+\epsilon}\right)\right)^\delta}, \end{aligned} \tag{4.9}$$

Using the preceding inequality and setting $x_0 = \lambda\delta(\log n)^{\delta-1}$ yields, for n large enough,

$$\begin{aligned}\log \mathbb{P}[N>n]^{-1} - \lambda(\log n)^\delta &\leqslant \lambda\left(\log n - \log\left(\frac{x_0}{1+\epsilon}\right)\right)^\delta - \lambda(\log n)^\delta + x_0 - \log(1-\epsilon) \\ &\leqslant -(1-\epsilon)\lambda\delta\,(\log n)^{\delta-1}\log\left(\lambda\delta(\log n)^{\delta-1}\right) + \lambda\delta(\log n)^{\delta-1},\end{aligned}$$

which, by passing $n\to\infty$ and then $\epsilon \to 0$, results in

$$\log \mathbb{P}[N>n]^{-1} - \lambda(\log n)^\delta \lesssim -\lambda\delta(\delta-1)(\log\log n)(\log n)^{\delta-1}. \tag{4.10}$$

Next, we prove the *upper bound*. Following the same approach as in the proof of Theorem 2.2, we obtain, for large n and $y = \lambda(\log n)^\delta - \lambda\delta(\delta-1)\log\log n(\log n)^{\delta-1}$,

$$\begin{aligned}\mathbb{P}[N>n] &\leqslant \sum_{k=0}^{y-1} e^{-k}\mathbb{P}\left[k \leqslant \frac{n}{\Phi^{\leftarrow}\left((1+\epsilon)V^{-1}\right)} \leqslant k+1\right] + e^{-y} + o\left(\mathbb{P}[N>n]\right) \\ &\leqslant (1+\epsilon)\sum_{k=0}^{y-1} e^{-k-\lambda(\log n - \log(k+1))^\delta} + e^{-y} + o\left(\mathbb{P}[N>n]\right).\end{aligned} \tag{4.11}$$

Suppose that $f(x) = x + \lambda\left(\log n - \log x\right)^\delta$ reaches the minimum at x^* when $1 \leqslant x \leqslant y$. It is easy to check that $f'(x) = 1 - \lambda\delta(\log n - \log x)^{\delta-1}/x$ is monotonically increasing for x in $(0,n)$. Then, by defining

$$x_1 \triangleq \lambda\delta(\log n)^{\delta-1} - (1-\epsilon)\lambda\delta(\delta-1)^2(\log\log n)(\log n)^{\delta-2}, \;\; \epsilon > 0,$$

we obtain, after some easy calculations, for large n,

$$f'(x_1) \geqslant 1 - \frac{(\log n)^{\delta-1} - (1-\epsilon/2)(\delta-1)^2(\log\log n)(\log n)^{\delta-2}}{(\log n)^{\delta-1} - (1-\epsilon)(\delta-1)^2(\log\log n)(\log n)^{\delta-2}} > 0,$$

which implies that $f'(x) > 0$ for $x \geqslant x_1$ and, therefore, $x^* < x_1$ for all $n > n_0$. Hence, by (4.11), we obtain

$$\mathbb{P}[N>n] \leqslant (1+\epsilon)ye^{1-\lambda(\log n - \log x_1)^\delta} + e^{-y} + o\left(\mathbb{P}[N>n]\right), \tag{4.12}$$

which, by recalling the definitions of y and x_1, results in

$$\log \mathbb{P}[N>n]^{-1} - \lambda(\log n)^\delta \gtrsim -(1+\epsilon)\lambda\delta(\delta-1)(\log\log n)(\log n)^{\delta-1}. \tag{4.13}$$

Finally, passing $\epsilon \to 0$ in (4.13) and combining it with (4.10), we finish the proof. □

4.5 Proof of Proposition 2.5

Proof: First, we prove the *lower bound*. Using the same arguments as in the proof of the lower bound for Theorem 2.3, we obtain, for $0 < \epsilon < 1, x_0 > 0$ and n large enough,

$$\log\left(\mathbb{P}[N > n]^{-1}\right) \leqslant x_0 + \frac{1}{1-\epsilon}\log\left(\Phi\left(\frac{(1+\epsilon)n}{x_0}\right)\right) = x_0 + \frac{1}{1-\epsilon}e^{\lambda\left(\log\left(\frac{(1+\epsilon)n}{x_0}\right)\right)^\delta}.$$

Setting $x_0 = e^{\lambda(\log n)^\delta\left(1-\delta\lambda(\log n)^{\delta-1}\right)}, 1/2 < \delta < 1$ in the preceding inequality yields

$$\log\left(\mathbb{P}[N > n]^{-1}\right) \leqslant e^{\lambda(\log n)^\delta\left(1-\delta\lambda(\log n)^{\delta-1}\right)} + \frac{1}{1-\epsilon}e^{\lambda(\log n - \log x_0 + \log(1+\epsilon))^\delta},$$

which, by noting that $\lambda(\log n - \log x_0 + \log(1+\epsilon))^\delta \leqslant \lambda(\log n)^\delta\left(1-(1-\epsilon)\delta\lambda(\log n)^{\delta-1}\right)$ for all n large enough, implies, for n large enough,

$$\log(\log \mathbb{P}[N > n]^{-1}) \leqslant \log\left(1 + \frac{1}{1-\epsilon}\right) + \lambda(\log n)^\delta\left(1-(1-\epsilon)\delta\lambda(\log n)^{\delta-1}\right).$$

Passing $\epsilon \to 0$ in the preceding inequality results in

$$\log(\log \mathbb{P}[N > n]^{-1}) - \lambda(\log n)^\delta \leqslant -\delta\lambda^2(\log n)^{2\delta-1}. \tag{4.14}$$

Next, we prove the *upper bound*. Following the same approach as in the proof of Theorem 2.2, we obtain

$$\mathbb{P}[N > n] \leqslant \sum_{k=0}^{y-1} e^{-k-\frac{1}{1+\epsilon}e^{\lambda(\log n - \log k)^\delta}} + e^{-y} + o\left(\mathbb{P}[N > n]\right). \tag{4.15}$$

Choose $y = e^{\lambda(\log n)^\delta\left(1-(1+\epsilon)\delta\lambda(\log n)^{\delta-1}\right)}$ and let $f(x) = x + e^{\lambda(\log n - \log x)^\delta}/(1+\epsilon)$. Since $f'(x) = 1 - e^{\lambda(\log n - \log x)^\delta}/((1+\epsilon)x)$ is an increasing function for x in $(0, n)$, it is easy to see that, for all $0 < x \leqslant y$ and n large enough,

$$f'(x) \leqslant 1 - \frac{e^{\lambda(\log n - \log y)^\delta}}{(1+\epsilon)y} \leqslant 1 - \frac{e^{\lambda(\log n)^\delta\left(1-\delta\lambda(\log n)^{\delta-1}\right)}}{(1+\epsilon)e^{\lambda(\log n)^\delta\left(1-(1+\epsilon)\delta\lambda(\log n)^{\delta-1}\right)}} < 0.$$

Therefore, for $0 \leqslant k \leqslant y$, we obtain

$$e^{-k-\frac{1}{1+\epsilon}e^{\lambda(\log n - \log k)^\delta}} \leqslant e^{-y-\frac{1}{1+\epsilon}e^{\lambda(\log n - \log y)^\delta}},$$

which, by (4.15), yields

$$\begin{aligned}\mathbb{P}[N > n] &\leqslant ye^{-y-\frac{1}{1+\epsilon}e^{\lambda(\log n - \log y)^\delta}} + e^{-y} + o\left(\mathbb{P}[N > n]\right)\\ &\leqslant (y+1)e^{-y} + o\left(\mathbb{P}[N > n]\right),\end{aligned} \tag{4.16}$$

implying

$$\log(\log \mathbb{P}[N > n]^{-1}) - \lambda(\log n)^\delta \gtrsim -(1+\epsilon)\delta\lambda^2(\log n)^{2\delta-1}. \tag{4.17}$$

Finally, by passing $\epsilon \to 0$ in (4.17) and combining it with (4.14), we finish the proof. □

4.6 Proof of Theorem 2.5

The proofs are based on large deviation results developed by S. V. Nagaev in [15]; specifically, we summarize Corollary 1.6 and Corollary 1.8 of [15] in this following lemma.

Lemma 4.1 *Let $X_1, X_2, \cdots X_n$ and X be i.i.d random variables with $\int_{u \geqslant 0} u^s d\mathbb{P}[X < u] < \infty$ and $\mathbb{E}X = 0$.*

If $1 \leqslant s \leqslant 2$, then, there exist finite $y_s, c > 0$ such that for $x > y > y_s$,

$$\mathbb{P}\left[\sum_{i=1}^{n} X_i \geqslant x\right] \leqslant n\mathbb{P}[X > y] + \left(\frac{cn}{xy^{s-1}}\right)^{x/2y}. \tag{4.18}$$

If $s > 2$, then, there exist finite $c > 0$ such that

$$\mathbb{P}\left[\sum_{i=1}^{n} X_i \geqslant x\right] \leqslant \frac{cn}{x^s} + \exp\left(\frac{-x^2}{cn}\right). \tag{4.19}$$

Proof: Please refer to [15].

Now, we are ready to prove Theorem 2.5.

Proof: First, we establish the *upper bound.* By recalling Definition 1.1, for any $1/2 > \delta > 0$, we obtain

$$\begin{aligned}
\mathbb{P}[T > t] &= \mathbb{P}\left[\sum_{i=1}^{N-1}(U_i + A_i) + L > t\right] \\
&\leqslant \mathbb{P}\left[\sum_{i=1}^{N}(A_i \wedge L + \mathbb{E}[U]) > (1-2\delta)t\right] + \mathbb{P}\left[\sum_{i=1}^{N}(U_i - \mathbb{E}[U]) > \delta t\right] + \mathbb{P}\left[L > \delta t\right] \\
&\triangleq I_1 + I_2 + I_3.
\end{aligned} \tag{4.20}$$

The condition $\mathbb{E}[L^{\alpha+\epsilon}] < \infty$ implies

$$I_3 \leqslant \frac{\mathbb{E}[L^{\alpha+\theta}]}{(\epsilon t)^{\alpha+\theta}} = O\left(\frac{1}{t^{\alpha+\theta}}\right). \tag{4.21}$$

For I_2, we begin with studying the case of $\alpha > 1$, i.e., when $\mathbb{E}[N] < \infty$. Since N is independent of $\{U_i\}$, by defining $X_i \triangleq U_i - \mathbb{E}[U_i]$, we obtain,

$$I_2 = \sum_{n=1}^{\infty} \mathbb{P}[N = n]\mathbb{P}\left[\sum_{i=1}^{n} X_i > \delta t\right].$$

To evaluate $\mathbb{P}\left[\sum_{i=1}^{n} X_i > \delta t\right]$ in the preceding equality, we need to apply Lemma 4.1, which results in two situations. If $1 < s \triangleq \alpha + \theta \leqslant 2$, using (4.18) with $y = \delta t/2$, we obtain, for all $n \geqslant 1$,

$$\mathbb{P}\left[\sum_{i=1}^{n} X_i > \delta t\right] \leqslant n\mathbb{P}[X_1 > \delta t/2] + \frac{2^{s-1}cn}{\delta^s t^s}, \tag{4.22}$$

implying

$$\begin{aligned}
I_2 &\leqslant \sum_{n=1}^{\infty} \mathbb{P}[N = n]\left(n\mathbb{P}[X_1 > \delta t/2] + \frac{2^{s-1}cn}{\delta^s t^s}\right) \\
&\leqslant \mathbb{E}[N]\mathbb{P}[X_1 > \delta t/2] + \frac{2^{s-1}c\mathbb{E}[N]}{\delta^s t^{\alpha+\theta}} = O\left(\frac{1}{t^{\alpha+\theta}}\right).
\end{aligned} \tag{4.23}$$

Otherwise, if $s = \alpha + \theta > 2$, by (4.19), we derive, for $0 < \delta < 1$, $0 < \gamma < \alpha\delta/(1+\delta)$,

$$\begin{aligned}
I_2 &\leqslant \mathbb{P}\left[\sum_{i=1}^{\lfloor t^{1+\delta}\rfloor} X_i > \delta t\right] + \mathbb{P}\left[N > t^{1+\delta}\right] \\
&= \sum_{n=1}^{\lfloor t^{1+\delta}\rfloor} \mathbb{P}[N=n]\mathbb{P}\left[\sum_{i=1}^{n} X_i > \delta t\right] + O\left(\frac{1}{t^{(1+\delta)(\alpha-\gamma)}}\right) \\
&\leqslant \frac{c\mathbb{E}[N]}{(\delta t)^{\alpha+\theta}} + \exp\left(-\frac{\delta^2 t^{1-\delta}}{c}\right) + O\left(\frac{1}{t^{(1+\delta)(\alpha-\gamma)}}\right),
\end{aligned}$$

which implies, for some $\nu > 0$,

$$I_2 = O\left(\frac{1}{t^{\alpha+\nu}}\right). \tag{4.24}$$

Now, we study the case when $0 < \alpha \leqslant 1$. For $1 < s \triangleq 1+\theta \leqslant 2, \theta > 0$, recalling (4.22) and noting that $\sum_{n=1}^{\lfloor t^{\zeta}\rfloor} n\mathbb{P}[N=n] \leqslant Ht^{\zeta(1-\alpha+\sigma)}$ for $\alpha > \theta > \sigma > 0, (\theta+1)/(\sigma+1) > \zeta > 1$, $H > 0$, we obtain, for some $\nu > 0$,

$$\begin{aligned}
I_2 &\leqslant \sum_{n=1}^{\lfloor t^{\zeta}\rfloor} \mathbb{P}[N=n]\left(n\mathbb{P}[X_1 > \delta t/2] + \frac{2^{s-1}cn}{\delta^s t^s}\right) + \mathbb{P}\left[N > t^{\zeta}\right] \\
&\leqslant Ht^{\zeta(1-\alpha+\sigma)}\left(\frac{\mathbb{E}\left[X_1^{1+\theta}\right]}{(\delta t/2)^{1+\theta}} + \frac{2^{s-1}c}{\delta^s t^{1+\theta}}\right) + \mathbb{P}\left[N > t^{\zeta}\right] \\
&= O\left(\frac{1}{t^{\alpha+\nu}}\right),
\end{aligned}$$

which, in conjunction with (4.23) and (4.24), yields, for some $\nu > 0$,

$$I_2 = O\left(\frac{1}{t^{\alpha+\nu}}\right). \tag{4.25}$$

Next, we study I_1. It is easy to obtain, for $\epsilon > 0$,

$$\begin{aligned}
I_1 &\leqslant \mathbb{P}\left[\sum_{i=1}^{\frac{(1-2\delta)t}{\mathbb{E}[A+U](1+\delta)}} (A_i \wedge (\epsilon t) + \mathbb{E}[U]) > (1-\delta)t\right] + \mathbb{P}\left[N > \frac{(1-2\delta)t}{\mathbb{E}[A+U](1+\delta)}\right] + \mathbb{P}[L > \epsilon t] \\
&\triangleq I_{11} + I_{12} + I_{13}.
\end{aligned} \tag{4.26}$$

By recalling Theorem 2.1, we know

$$\mathbb{P}\left[N > \frac{(1-2\delta)t}{\mathbb{E}[A+U](1+\delta)}\right] \sim \frac{\Gamma(\alpha+1)(\mathbb{E}[U+A](1+\delta))^{\alpha}}{\Phi((1-2\delta)t)}. \tag{4.27}$$

The same argument for (4.21) implies

$$I_{13} = O\left(\frac{1}{t^{\alpha+\theta}}\right). \tag{4.28}$$

Furthermore, I_{11} is upper bounded by

$$\mathbb{P}\left[\sum_{i=1}^{\frac{(1-2\delta)t}{\mathbb{E}[A+U](1+\delta)}}(A_i\wedge(\epsilon t)+\mathbb{E}[U])-(1+\delta)\mathbb{E}[A+U]\frac{(1-2\delta)t}{\mathbb{E}[A+U](1+\delta)}>\delta t\right]$$
$$\leqslant \mathbb{P}\left[\sup_n\left\{\sum_{i=1}^{n}(A_i\wedge(\epsilon t)+\mathbb{E}[U])-n(1+\delta)\mathbb{E}[A+U]\right\}>\delta t\right],$$

where in the preceding probability, $\sup_n\{\sum_{i=1}^{n}(A_i\wedge(\epsilon t)+\mathbb{E}[U])-n(1+\delta)\mathbb{E}[A+U]\}$ is equal in distribution to the stationary workload in a $D/GI/1$ queue with truncated service times with the stability condition $\mathbb{E}[(A\wedge(\epsilon t)+\mathbb{E}[U])]<(1+\delta)\mathbb{E}[A+U]$. Therefore, using a similar proof for Lemma 3.2 of [6], we can show that for any $\beta>0$, there exists $\epsilon>0$ such that

$$I_{11}=o\left(\frac{1}{t^{\beta}}\right),$$

which, in conjunction with (4.27), (4.28), (4.26), and (4.20), (4.21), (4.25), yields, by passing $\epsilon,\delta\to 0$ in (4.27),

$$\mathbb{P}\left[T>t\right]\lesssim\frac{\Gamma(\alpha+1)(\mathbb{E}[U+A])^{\alpha}}{\Phi(t)}. \tag{4.29}$$

Then, we prove the *lower bound*. It is easy to obtain, for $\delta>0$,

$$\begin{aligned}\mathbb{P}[T>t]&=\mathbb{P}\left[\sum_{i=1}^{N-1}(U_i+A_i)+L>t\right]\\&\geqslant\mathbb{P}\left[\sum_{i=1}^{N-1}(U_i+A_i)>t, N\geqslant\frac{t(1+\delta)}{\mathbb{E}[U+A]}+1\right]\\&\geqslant\mathbb{P}\left[N\geqslant\frac{t(1+\delta)}{\mathbb{E}[U+A]}+1\right]-\mathbb{P}\left[\sum_{i=1}^{N-1}(U_i+A_i)\leqslant t, N\geqslant\frac{t(1+\delta)}{\mathbb{E}[U+A]}+1\right]\\&\triangleq I_1-I_2.\end{aligned}\tag{4.30}$$

For I_2, by defining $Y_i\triangleq U_i+A_i-\mathbb{E}[U+A]$, we obtain

$$I_2\leqslant\mathbb{P}\left[\sum_{i\leqslant t(1+\delta)/\mathbb{E}[U+A]}(U_i+A_i)\leqslant t\right]=\mathbb{P}\left[\sum_{i\leqslant t(1+\delta)/\mathbb{E}[U+A]}(-Y_i)\geqslant\delta t\right]$$

with $(-Y_i)\leqslant\mathbb{E}[U+A]<\infty$. By Chernoff bound, there exists $h,\eta>0$, such that

$$I_2\leqslant O\left(he^{-\eta t}\right), \tag{4.31}$$

which, by Theorem 2.1, equation (4.30) and passing $\delta\to 0$, yields

$$\mathbb{P}\left[T>t\right]\gtrsim\frac{\Gamma(\alpha+1)(\mathbb{E}[U+A])^{\alpha}}{\Phi(t)}. \tag{4.32}$$

Combining (4.29) and (4.32) completes the proof. □

4.7 Proof of Theorem 2.6

Proof: First, we prove the *upper bound*. It is easy to see that for $0<\epsilon<1$,

$$\begin{aligned}
\mathbb{P}[T>(1+\epsilon)t] &= \mathbb{P}\left[\sum_{i=1}^{N-1}((A_i\wedge L)+U_i)+L>(1+\epsilon)t\right]\\
&\leqslant \mathbb{P}\left[\sum_{i=1}^{\lceil t/l(t)\rceil}(A_i\wedge L)>\frac{t}{2}\right]+\mathbb{P}\left[\sum_{i=1}^{\lceil t/l(t)\rceil}U_i>\frac{t}{2}\right]+\mathbb{P}\left[N>\left\lceil\frac{t}{l(t)}\right\rceil+1\right]\\
&\quad+\mathbb{P}[L>\epsilon t]\\
&\triangleq I_1+I_2+I_3+I_4. \qquad (4.33)
\end{aligned}$$

Now, since $l(\cdot)$ is slowly varying and $\mathbb{P}[L>x]=O(\Phi(x)^{-1})$, we obtain,

$$I_4=\mathbb{P}[L>t]=o\left(\Phi(t)^{1-\epsilon}\right). \qquad (4.34)$$

By Theorem 2.2, we obtain

$$\lim_{t\to\infty}\frac{\log\left(\mathbb{P}\left[N>\left\lceil\frac{t}{l(t)}\right\rceil+1\right]^{-1}\right)}{\log\Phi\left(\frac{t}{l(t)}\right)}=1,$$

which, by (2.19), yields

$$\lim_{t\to\infty}\frac{\log\left(\mathbb{P}\left[N>\left\lceil\frac{t}{l(t)}\right\rceil+1\right]^{-1}\right)}{\log\left(\Phi(t)\right)}=1. \qquad (4.35)$$

Next, we evaluate I_1 and I_2. For I_2,

$$\begin{aligned}
I_2 &= \mathbb{P}\left[\sum_{i=1}^{\lceil t/l(t)\rceil}U_i>\frac{t}{2}\right]\\
&\leqslant \left\lceil\frac{t}{l(t)}\right\rceil\mathbb{P}\left[U_1>\frac{t}{l(t)}\right]+\mathbb{P}\left[\sum_{i=1}^{\lceil t/l(t)\rceil}U_i\wedge\frac{t}{l(t)}>\frac{t}{2}\right]\\
&\triangleq I_{21}+I_{22}. \qquad (4.36)
\end{aligned}$$

For $\delta>0$ and large t, due to condition (2.43) we obtain $l(t/l(t))\geqslant(1-\delta/2)l(t)$, which yields

$$I_{21}\leqslant O\left(e^{-(1+\delta)l\left(\frac{t}{l(t)}\right)}\right)\leqslant O\left(e^{-(1+\delta)(1-\frac{\delta}{2})l(t)}\right)=o\left(\Phi(t)^{-1}\right). \qquad (4.37)$$

Then, by using Chernoff bound, for $h>0$, we obtain

$$\begin{aligned}
I_{22} &= \mathbb{P}\left[e^{h\left(\sum_{i=1}^{\lceil t/l(t)\rceil}X_i\wedge\frac{t}{l(t)}\right)}>e^{ht/2}\right]\\
&\leqslant e^{-\frac{ht}{2}}\left(\mathbb{E}\left[e^{h\left(X_i\wedge\frac{t}{l(t)}\right)}\right]\right)^{\frac{t}{l(t)}+1},
\end{aligned}$$

which, by selecting $h=4l(t)/t$, and noting that

$$e^{h\left(X_i\wedge\frac{t}{l(t)}\right)}\leqslant 1+(e^4-1)\frac{l(t)}{t}\left(X_1\wedge\frac{t}{l(t)}\right),$$

implies

$$\begin{aligned}
I_{22} &\leqslant e^{-2l(t)} \left(\mathbb{E}\left[1 + (e^4 - 1)\frac{l(t)}{t} \left(X_1 \wedge \frac{t}{l(t)} \right) \right] \right)^{\frac{t}{l(t)}+1} \\
&\leqslant e^{-2l(t)} \left(1 + (e^4 - 1)\frac{l(t)}{t} \mathbb{E}\left[X_1 \right] \right)^{\frac{t}{l(t)}+1} \\
&= o\left(\frac{1}{\Phi(t)} \right). \qquad (4.38)
\end{aligned}$$

Combining (4.36), (4.37) and (4.38) yields

$$I_2 = o\left(\frac{1}{\Phi(t)} \right). \qquad (4.39)$$

For I_1, it is easy to see

$$\begin{aligned}
I_1 &= \mathbb{P}\left[\sum_{i=1}^{\lceil t/l(t) \rceil} (A_i \wedge L) > \frac{t}{2} \right] \\
&\leqslant \mathbb{P}\left[\sum_{i=1}^{\lceil t/l(t) \rceil} \left(A_i \wedge \frac{t}{l(t)} \right) > \frac{t}{l(t)} \right] + \mathbb{P}\left[L > \frac{t}{l(t)} \right] \\
&\triangleq I_{11} + I_{12}. \qquad (4.40)
\end{aligned}$$

Using the same argument as in deriving (4.38), we can prove that $I_{11} = o(1/\Phi(t))$, which, by noting condition (2.43) implying $I_{12} = o(1/\Phi(t))$, yields

$$I_1 = o\left(\frac{1}{\Phi(t)} \right). \qquad (4.41)$$

Combining (4.33), (4.34), (4.35), (4.39) and (4.41), yields

$$\overline{\lim_{t\to\infty}} \frac{\log \mathbb{P}[T > t]}{\log(\Phi(t))} \leqslant -1. \qquad (4.42)$$

Next, we prove the *lower bound*. Observe

$$\begin{aligned}
\mathbb{P}\left[\sum_{i=1}^{N-1} (A_i + U_i) + L > t \right] &\geqslant \mathbb{P}\left[\sum_{i=0}^{N-1} (A_i \wedge 1) > t, N > \left\lceil \frac{2t}{\mathbb{E}[A \wedge 1]} \right\rceil + 1 \right] \\
&\geqslant \mathbb{P}\left[N > \left\lceil \frac{2t}{\mathbb{E}[A \wedge 1]} \right\rceil + 1 \right] - \mathbb{P}\left[\sum_{i=1}^{\left\lceil \frac{2t}{\mathbb{E}[A\wedge 1]} \right\rceil} (A_i \wedge 1) \leqslant t \right],
\end{aligned}$$

and, by using the same arguments as in deriving (4.31), it is very easy to prove that the second probability on the right hand side of the second inequality above is exponentially bounded. Therefore, using Theorem 2.2 and the preceding exponential bound yields

$$\underline{\lim_{t\to\infty}} \frac{\log \mathbb{P}[T > t]}{\Phi(\log t)} \geqslant -1. \qquad (4.43)$$

Combining (4.42) and (4.43) completes the proof. □

4.8 Proof of Theorem 2.7

Proof: First, we prove the upper bound. It is easy to see that, for $\eta \triangleq \mathbb{E}[U]/\mathbb{E}[A+U]$ and $0 < \epsilon < 1$,

$$\begin{aligned}\mathbb{P}[T > (1+\epsilon)t] &= \mathbb{P}\left[\sum_{i=1}^{N-1}((A_i \wedge L) + U_i) + L > t\right] \\ &\leqslant \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]}\right\rfloor}(A_i \wedge L) > (1-\eta)t\right] + \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]}\right\rfloor} U_i > \eta t\right] \\ &\quad + \mathbb{P}\left[N > \left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]}\right\rfloor\right] + \mathbb{P}[L > \epsilon t] \\ &\triangleq I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.44}$$

The condition on L implies

$$I_4 = \mathbb{P}[L > \epsilon t] = o\left(e^{-(\log \Phi(x))^{1/(\beta+1)}}\right), \tag{4.45}$$

and, by Theorem 2.3, we obtain

$$\lim_{t\to\infty} \frac{\log\left(\mathbb{P}\left[N > \left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[X_1]}\right\rfloor\right]^{-1}\right)}{(\log \Phi(t))^{\frac{1}{\beta+1}}} = (1-\epsilon)^{\frac{\beta}{\beta+1}} \frac{\beta^{\frac{1}{\beta+1}} + \beta^{-\frac{\beta}{\beta+1}}}{(\mathbb{E}[A+U])^{\frac{\beta}{\beta+1}}}. \tag{4.46}$$

Now, we evaluate I_2. By applying the large deviation result proved in Theorem 3.2 (ii) of [6], and noting $\mathbb{P}[U > x] \leqslant o\left(e^{-x^{(1+\delta/2)\beta/(\beta+1)}}\right)$, we can prove that there exist $1 > \gamma > 0$ and $C > 0$, such that

$$\begin{aligned}\mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]}\right\rfloor}(U_i \wedge \gamma\epsilon\eta t) - \eta(1-\epsilon)t > \epsilon\eta t\right] &\leqslant C\left(e^{-(\epsilon\eta t)^{(1+\delta/2))\beta/(\beta+1)}}\right) \\ &= o\left(e^{-(\log \Phi(x))^{1/(\beta+1)}}\right). \end{aligned} \tag{4.47}$$

Thus, considering I_2, we obtain

$$I_2 \leqslant \left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[X_1]}\right\rfloor \mathbb{P}\left[U_1 > (\gamma\epsilon\eta)t\right] + \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[X_1]}\right\rfloor}(U_i \wedge \gamma\epsilon\eta t) > \eta t\right], \tag{4.48}$$

which, by (4.47) and the assumption on U, yields,

$$I_2 = o\left(e^{-(\log \Phi(x))^{1/(\beta+1)}}\right). \tag{4.49}$$

For I_1, we begin with proving the situation when $\zeta = 0$, $\xi > \beta$, i.e., assuming no conditions

on $\mathbb{P}[A > x]$ beyond $\mathbb{E}[A] < \infty$. It is easy to obtain, for $0 < \epsilon < 1/(\beta+1)$,

$$\begin{aligned} I_1 &= \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor} (A_i \wedge L) > (1-\eta)t\right] \\ &\leqslant \mathbb{P}\left[L > t^{\frac{1}{\beta+1}-\epsilon}\right] + \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor} \left(A_i \wedge t^{\frac{1}{\beta+1}-\epsilon}\right) > (1-\eta)t\right] \\ &\triangleq I_{11} + I_{12}. \end{aligned} \tag{4.50}$$

The condition $\xi > \beta$ implies, for $0 < \epsilon < (1-\beta/\xi)/(\beta+1)$,

$$I_{11} \leqslant O\left(e^{-t^{\left(\frac{1}{\beta+1}-\epsilon\right)\xi}}\right) = o\left(e^{-(\log \Phi(t))^{1/(\beta+1)}}\right). \tag{4.51}$$

And, by using Chernoff bound, for $h > 0$, we obtain

$$I_{12} = \mathbb{P}\left[e^{h\left(\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A_1]} \right\rfloor}\left(A_i \wedge t^{\frac{1}{\beta+1}-\epsilon}\right)\right)} > e^{h(1-\eta)t}\right] \leqslant e^{-h(1-\eta)t}\left(\mathbb{E}\left[e^{h\left(A_1 \wedge t^{\left(\frac{1}{\beta+1}-\epsilon\right)}\right)}\right]\right)^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A_1]} \right\rfloor},$$

which, by selecting $h = \epsilon(1-\eta)t^{-\left(\frac{1}{\beta+1}-\epsilon\right)}$, and using $e^x \leqslant 1 + (e^b - 1)x/b$ for $0 \leqslant x \leqslant b$, yields

$$e^{h\left(A_1 \wedge t^{\left(\frac{1}{\beta+1}-\epsilon\right)}\right)} \leqslant 1 + \frac{e^{\epsilon(1-\eta)} - 1}{\epsilon(1-\eta)} h\left(A_1 \wedge t^{\left(\frac{1}{\beta+1}-\epsilon\right)}\right).$$

Then, the preceding inequalities, for ϵ small enough such that $\epsilon(1-\eta) - (1-\epsilon)(e^{\epsilon(1-\eta)} - 1) > 0$, imply

$$\begin{aligned} I_{12} &\leqslant e^{-\epsilon(1-\eta)t^{\frac{\beta}{\beta+1}+\epsilon}}\left(\mathbb{E}\left[1 + \frac{e^{\epsilon(1-\eta)} - 1}{\epsilon(1-\eta)} h\left(A_1 \wedge t^{\left(\frac{1}{\beta+1}-\epsilon\right)}\right)\right]\right)^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A_1]} \right\rfloor} \\ &\leqslant e^{-\epsilon(1-\eta)t^{\frac{\beta}{\beta+1}+\epsilon}}\left(1 + (e^{\epsilon(1-\eta)} - 1)\frac{t^{\frac{\beta}{\beta+1}+\epsilon}}{t}\mathbb{E}\left[A_1\right]\right)^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A_1]} \right\rfloor} \\ &= O\left(e^{-\left(\epsilon(1-\eta)-(1-\epsilon)(e^{\epsilon(1-\eta)}-1)\right)t^{\frac{\beta}{\beta+1}+\epsilon}}\right) = o\left(e^{-(\log \Phi(t))^{1/(\beta+1)}}\right). \end{aligned} \tag{4.52}$$

Combining (4.51) and (4.52) yields $I_1 = o\left(e^{-(\log \Phi(t))^{1/(\beta+1)}}\right)$ for $\zeta = 0, \xi > \beta$.

Now, in order to prove the situation $\zeta > 0$ when $\mathbb{P}[A > x]$ is bounded by a Weibull distribution, we need to use the following lemma that is based on a minor modification of Theorem 3.2 (ii) in [6] (or Lemma 2 in [7]) that can be proved by selecting $s = \upsilon Q(u)/u, 0 < \upsilon < 1$ in (5.18) of [6], where $Q(u)$ is defined in [6].

Lemma 4.2 *If $\mathbb{P}[A > x] \leqslant He^{-x^\zeta}$, $H > 0, 1 > \zeta > 0$, then, for $x^\theta < u < \epsilon x$, $\epsilon > 0$, $1 > \theta > 0$ and $n \leqslant Hx$, there exist $C > 0, 1 > \delta > 0$, such that*

$$\mathbb{P}\left[\sum_{i=1}^{n} A_i \wedge u - n\mathbb{E}[A] > x\right] \leqslant Ce^{-\delta u^{\zeta-1}x}.$$

Note that the case $\zeta \geqslant 1$ is trivial since in this situation I_1 is exponentially bounded using Chernoff bound. Therefore, we only need to consider the situation $0 < \zeta < 1$. Using the union bound and the independence of $\{A_i\}$ and L, it is easy to obtain, for $0 < \epsilon < 1/(\beta+1)$,

$$\begin{aligned}
I_1 &= \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor} (A_i \wedge L) > (1-\eta)t\right] \\
&\leqslant \mathbb{P}\left[L > \epsilon t\right] + \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor} \left(A_i \wedge t^{\frac{1}{\beta+1}-\epsilon}\right) > (1-\eta)t\right] \\
&\quad + \int_{t^{\frac{1}{\beta+1}-\epsilon}}^{\epsilon t} \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor} (A_i \wedge u) > (1-\eta)t\right] d\mathbb{P}[L \leqslant u] \\
&\triangleq I_{11} + I_{12} + I_{13}. \qquad (4.53)
\end{aligned}$$

From (4.45) and (4.52), we obtain

$$I_{11} + I_{12} = o\left(e^{-(\log \Phi(t))^{1/(\beta+1)}}\right). \qquad (4.54)$$

Applying Lemma 4.2 yields, for $t^{1/(\beta+1)-\epsilon} \leqslant u \leqslant \epsilon t$,

$$\begin{aligned}
\mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor} (A_i \wedge u) > (1-\eta)t\right] &= \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor} (A_i \wedge u) - \left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor \mathbb{E}[A] > \epsilon(1-\eta)t\right] \\
&\leqslant Ce^{-\delta\epsilon(1-\eta)u^{\zeta-1}t},
\end{aligned}$$

resulting in, for some $h > 0$,

$$\begin{aligned}
I_{13} &\leqslant \int_{t^{\frac{1}{\beta+1}-\epsilon}}^{\epsilon t} Ce^{-\delta\epsilon(1-\eta)tu^{\zeta-1}} d\mathbb{P}[L \leqslant u] \\
&\leqslant Ce^{-\delta\epsilon(1-\eta)tu^{\zeta-1}} \mathbb{P}[L > u] \Big|_{\epsilon t}^{t^{\frac{1}{\beta+1}-\epsilon}} + \int_{t^{\frac{1}{\beta+1}-\epsilon}}^{\epsilon t} He^{-u^{\xi}} Ce^{-\delta\epsilon(1-\eta)tu^{\zeta-1}} (1-\zeta)\delta\epsilon(1-\eta)tu^{\zeta-2} du \\
&\leqslant \sup_{t^{\frac{1}{\beta+1}-\epsilon} \leqslant u \leqslant \epsilon t} \left\{Ce^{-u^{\xi}-\delta\epsilon(1-\eta)tu^{\zeta-1}}\right\} \left(1 + \int_{t^{\frac{1}{\beta+1}-\epsilon}}^{\epsilon t} H(1-\zeta)\delta\epsilon(1-\eta)tu^{\zeta-2} du\right) \\
&= O\left(e^{-ht^{\xi/(\xi+1-\zeta)}}\right) = o\left(e^{-(\log \Phi(t))^{1/(\beta+1)}}\right).
\end{aligned}$$

The preceding bound on I_{13}, in conjunction with (4.54) and the proof of the case for $\zeta = 0$, implies, for all $\zeta \geqslant 0$,

$$I_1 = o\left(e^{-(\log \Phi(t))^{1/(\beta+1)}}\right). \qquad (4.55)$$

Thus, combining (4.44), (4.45), (4.46), (4.49), (4.55), and passing $\epsilon \to 0$ yields

$$\lim_{t \to \infty} \frac{\log \mathbb{P}[T > t]^{-1}}{(\log \Phi(t))^{\frac{1}{\beta+1}}} \geqslant \frac{\beta^{\frac{1}{\beta+1}} + \beta^{-\frac{\beta}{\beta+1}}}{(\mathbb{E}[A+U])^{\frac{\beta}{\beta+1}}}. \qquad (4.56)$$

Now, we prove the *lower bound*. Using the same argument as in deriving equation (4.30) in the proof of the lower bound for Theorem 2.5, it is easy to obtain, for $\delta > 0$,

$$\mathbb{P}[T > t] \geqslant \mathbb{P}\left[N \geqslant \frac{t(1+\delta)}{\mathbb{E}[U+A]} + 1\right] - \mathbb{P}\left[\sum_{i=1}^{N-1}(U_i + A_i) \leqslant t, N \geqslant \frac{t(1+\delta)}{\mathbb{E}[U+A]} + 1\right],$$

where the second probability on the right hand side of the preceding inequality is exponentially bounded (see (4.31)). Therefore, using Theorem 2.3 and passing $\delta \to 0$ yields

$$\lim_{t\to\infty} \frac{\log \mathbb{P}[T > t]^{-1}}{(\log \Phi(t))^{\frac{1}{\beta+1}}} \leqslant \frac{\beta^{\frac{1}{\beta+1}} + \beta^{-\frac{\beta}{\beta+1}}}{(\mathbb{E}[A+U])^{\frac{\beta}{\beta+1}}}. \tag{4.57}$$

Combining (4.56) and (4.57) completes the proof. □

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Characterizing Heavy-Tailed Distributions Induced by Retransmissions

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Abstract

Consider a generic data unit of random size L that needs to be transmitted over a channel of unit capacity. The channel availability dynamics is modeled as an i.i.d. sequence $\{A, A_i\}_{i\geqslant 1}$ that is independent of L. During each period of time that the channel becomes available, say A_i, we attempt to transmit the data unit. If $L \leq A_i$, the transmission is considered successful; otherwise, we wait for the next available period A_{i+1} and attempt to retransmit the data from the beginning. We investigate the asymptotic properties of the number of retransmissions N and the total transmission time T until the data is successfully transmitted. In the context of studying the completion times in systems with failures where jobs restart from the beginning, it was first recognized in [**?**, **?**] that this model results in power law and, in general, heavy-tailed delays. The main objective of this paper is to uncover the detailed structure of this class of heavy-tailed distributions induced by retransmissions.

More precisely, we study how the functional dependence $(\mathbb{P}[L > x])^{-1} \approx \Phi((\mathbb{P}[A > x])^{-1})$ impacts the distributions of N and T; the approximation $\approx$ will be appropriately defined in the paper depending on the context. In the functional space of $\Phi(\cdot)$, we discover several functional criticality points that separate classes of different functional behavior of the distribution of N. For example, we show that if $\log(\Phi(n))$ is slowly varying, then $\log(\mathbb{P}[N > n])$ is essentially slowly varying as well. Interestingly, if $\log(\Phi(n))$ grows slower than $e^{\sqrt{\log n}}$ then we have the asymptotic equivalence $\log(\mathbb{P}[N > n]) \approx -\log(\Phi(n))$. However, if $\log(\Phi(n))$ grows faster than $e^{\sqrt{\log n}}$, this asymptotic equivalence does not hold and admits a different functional form. Similarly, different types of functional behavior are shown for moderately heavy tails (Weibull distributions) where $\log(\mathbb{P}[N > n]) \approx -(\log \Phi(n))^{1/(\beta+1)}$ assuming $\log \Phi(n) \approx n^{\beta}$, as well as the nearly exponential ones of the form $\log(\mathbb{P}[N > n]) \approx -n/(\log n)^{1/\gamma}, \gamma > 0$ when $\Phi(\cdot)$ grows faster than two exponential scales $\log\log(\Phi(n)) \approx n^{\gamma}$.

We also discuss the engineering implications of our results on communication networks since retransmission strategy is a fundamental component of the existing network protocols on all communication layers, from the physical to the application one.

Keywords: Retransmissions, Channel (systems) with failures, Restarts, Origins of heavy-tails (subexponentiality), Gaussian distributions, Exponential distributions, Weibull distributions, Log-normal distributions, Power laws.

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1 Introduction

Retransmissions represent one of the most fundamental approaches in communication networks that guarantee data delivery in the presence of channel failures. These types of mechanisms have been employed on all networking layers, including, for example, Automatic Repeat reQuest (ARQ) protocol (e.g., see Section 2.4 of [?]) in the data link layer where a packet is resent automatically in case of an error; contention based ALOHA type protocols in the medium access control (MAC) layer that use random backoff and retransmission mechanism to recover data from collisions; end-to-end acknowledgement for multi-hop transmissions in the transport layer; HTTP downloading scheme in the application layer, etc. We discuss the engineering implications of our results at the end of this introduction and, in more detail, in Section 3.

As briefly stated in the abstract, we use the following generic channel with failures [?] to model the preceding situations. The channel dynamics is described as an on-off process $\{(A,U),(A_i,U_i)\}_{i\geqslant 1}$ with alternating periods when channel is available A_i and unavailable U_i, respectively; $(A,A_i)_{i\geqslant 1}$ and $(U,U_i)_{i\geqslant 1}$ are two independent sequences of i.i.d random variables. In each period of time that the channel becomes available, say A_i, we attempt to transmit the data unit of random size L. If $L\leq A_i$, we say that the transmission is successful; otherwise, we wait for the next period A_{i+1} when the channel is available and attempt to retransmit the data from the beginning. We study the asymptotic properties of the distributions of the total transmission time T and number of retransmissions N, for the precise definitions of these variables and the model, see the following Subsection 1.1.

The preceding model was introduced and studied in [?] and, apart from the already mentioned applications in communications, it represents a generic model for other situations where jobs have to restart from the beginning after a failure. It was first recognized in [?] that this model results in power law distributions when the distributions of L and A have a matrix exponential representation, and this result was rigorously proved and further generalized in [?]. Under more general conditions, [?] discovers that the distributions of N and T follow power laws with the same exponent α as long as $\log\mathbb{P}[L>x]\approx\alpha\log\mathbb{P}[A>x]$ for large x, which implies that power law distributions, possibly with infinite mean ($0<\alpha<1$) and variance ($0<\alpha<2$), may arise even when transmitting superexponential (e.g., Gaussian) documents/packets. More recent results on the heavy-tailed completion times in a system with failures are developed in [?]. In this paper, we further characterize this class of heavy-tailed distributions that are induced by retransmissions.

Technically speaking, our proofs are based on the method introduced in [?] that uses the following key arguments. First, in exploring the distribution of N, we assume that the functional relationship $\Phi(\cdot)$, with $\bar{F}^{-1}(x)\approx\Phi(\bar{G}^{-1}(x))$ between the probability distributions of $\bar{F}(x)\triangleq\mathbb{P}[L>x]$ and $\bar{G}(x)\triangleq\mathbb{P}[A>x]$, is eventually monotonically increasing, which guarantees the existence of an asymptotic inverse $\Phi^{\leftarrow}(\cdot)$ of $\Phi(\cdot)$, and then, we use the result that $\bar{F}(L)$ is a uniform random variable on $(0,1)$ given that $\bar{F}(\cdot)$ is continuous (see [?, ?]), e.g., for $\bar{F}(x)=\left(\bar{G}(x)\right)^{\alpha},\alpha>0$, the key argument on the uniform distribution of $\bar{F}(L)$ from [?] can be illustrated as

$$\mathbb{P}[N>n]=\mathbb{E}\left[(1-\bar{G}(L))^n\right]\approx\mathbb{E}[e^{-n\bar{G}(L)}]=\mathbb{E}\left[e^{-n\bar{F}^{1/\alpha}(L)}\right]=\frac{\Gamma(\alpha+1)}{n^{\alpha}}.$$

Second, in contrast to [?, ?], instead of studying the total transmission time T directly, we study a simpler quantity N and then use the large deviations technique to investigate T, since T can be represented as a sum of L and $\{(A_i+U_i)\}_{1\leqslant i<N}$; see equation (1.1) in the next subsection. Hence, our analysis is entirely probabilistic, which differs from the work in [?] that relies on Tauberian theorems.

More precisely, we extend the results from [**?**, **?**] under a more unified framework and study how the functional dependence between the data characteristics and channel dynamics in the form $(\mathbb{P}[L > x])^{-1} \approx \Phi(\mathbb{P}[A > x])^{-1})$ impacts the distribution of N, where the approximation $\approx$ will be possibly differently defined according to the context. In the functional space of $\Phi(n)$, we identify several functional criticality points that define different classes of functional behavior of the distribution of N. Specifically, in Subsection 2.1.1, we show that if $\Phi(n)$ is dominantly varying, e.g., regularly varying, then $\mathbb{P}[N > n] \approx \Phi(n)^{-1}$; see Proposition 2.2 and Theorem 2.1. As shown in Proposition 2.3, the preceding tail equivalence between $\mathbb{P}[N > n]$ and $\Phi(n)^{-1}$ basically does not hold if $\Phi(x)$ is not dominantly varying, e.g., if $\Phi(x)$ is lognormal. Furthermore, we show in a weaker form that if $\log(\Phi(n))$ is slowly varying, then $\log\left((\mathbb{P}[N > n])^{-1}\right)$ is essentially slowly varying as well, as proved in Proposition 2.1. Interestingly, if $\log(\Phi(n))$ grows slower than $e^{\sqrt{\log n}}$ then we have the asymptotic equivalence $\log\left(\mathbb{P}[N > n]\right) \approx -\log(\Phi(n))$ as shown in Theorem 2.2 and Corollary 2.3, which implies parts (1:1), (2:1) and (2:2) of Theorem 2.1 in [**?**] and extends Theorem 2 in [**?**]. However, if $\log(\Phi(n))$ grows faster than $e^{\sqrt{\log n}}$, this asymptotic equivalence does not hold and we demonstrate a different functional form in Proposition 2.5.

Next, for lighter distributions of Weibull type, in Subsection 2.1.2, we show that if $\log(\Phi(n))$ is regularly varying with index $\beta > 0$, then basically one obtains Weibull distribution for N, i.e., $\log\left(\mathbb{P}[N > n]\right) \approx -\left(\log \Phi(n)\right)^{1/(\beta+1)}$, as shown in Theorem 2.3, which we term moderately heavy (Weibull tail) asymptotics; this result implies part (1:2) of Theorem 2.1 in [**?**], and provides a more precise logarithmic asymptotics instead of a double logarithmic limit. Finally, in Subsection 2.1.3, we consider the situation when the separation between $\mathbb{P}[L > x]$ and $\mathbb{P}[A > x]$ is very large, i.e., their distributions are roughly separated by more than two exponential scales ($\log\log\left(\Phi(n)\right) \approx n^{\gamma}$). This separation results in what we call the nearly exponential distribution for N in the form $\log\left(\mathbb{P}[N > n]\right) \approx -n/(\log n)^{1/\gamma}$.

After the preceding characterization of the different classes of distributional behavior for N, we study in Subsection 2.2 the total transmission time T. As previously stated for studying T, we use the large deviation results since T can be represented as the sum of L and $\{(A_i + U_i\}_{1 \leqslant i < N}$. In this context, our primary results show that: (i) when $\Phi(\cdot)$ is regularly varying, we derive the exact asymptotics for T in Theorem 2.5. (ii) when $\log(\Phi(\cdot))$ is slowly varying, we obtain the logarithmic asymptotics for T in Theorem 2.6. (iii) when $\log(\Phi(\cdot))$ is regularly varying with positive index, we derive, in a different scale than in Theorem 2.6, the logarithmic asymptotics in Theorem 2.7. Note that the preceding three results on T correspond to Theorems 2.1 i), 2.2 and 2.3 on N, respectively. Similarly, one can derive the respective statements on $\mathbb{P}[T > t]$ for other results on $\mathbb{P}[N > n]$, but we omit this to avoid lengthy expositions and repetitions. Interestingly, we want to point out that, unlike Theorems 2.5 and 2.6 requiring no conditions on A (Theorem 2.5 needs $\mathbb{E}[A] < \infty$), the minimum conditions needed for Theorem 2.7, as shown by Proposition 2.7, basically involve a balance between the tail decays of $\mathbb{P}[A > x]$ and $\mathbb{P}[L > x]$.

From a practical perspective, our results suggest that careful examination and possible redesign of retransmission based protocols in communication networks might be needed. This is especially the case for Ad Hoc and resource limited sensor networks, where frequent channel failures occur due to a variety of reasons, including signal fading, multipath effects, interference, contention with other nodes, obstructions, node mobility, and other changes in the environment [**?**]. In engineering applications, our main discovery is the matching between the statistical characteristics of the channel and transmitted data (packets). On the network application layer, most of us have been inconvenienced when the connections would brake while we are downloading a large file from the Internet. This issue has been already recognized in

practice where software for downloading files was developed that would save the intermediate data (checkpoints) and resume the download from the point when the connection was broken. However, our results emphasize that, in the presence of frequently failing connections, the long delays may arise even when downloading relatively small documents. Hence, we argue that one might need to modify the application layer software, especially for the wireless environment, by introducing checkpoints even for small to moderate size documents. In our related papers, we found that several well-known retransmission based protocols in different layers of networking architecture can lead to power law delays, e.g., ALOHA type protocols in MAC layer [?] and end-to-end acknowledgements in transport layer [?]. These new findings suggest that special care should be taken when designing robust networking protocols, especially in the wireless environment where channel failures are frequent. We discuss these and other engineering implications of our results in Section 3.

We also discuss possible solutions to alleviate this problem, such as assigning checkpoints, breaking large packets into smaller units preferably by using dynamic packet fragmentation techniques [?]. Clearly, there is a tradeoff between the sizes of these newly created packets and the throughput since, if the packets are too small, they will mostly contain the packet headers and, thus, very little useful information.

Finally, we would like to point out that, in addition to the preceding applications in communication networks [?, ?, ?, ?] and job processing on machines with failures [?, ?], the model studied in this paper may represent a basis for understanding more complex failure prone systems, e.g., see the recent study on parallel computing in [?].

The rest of the paper is organized as follows. After a detailed description of the channel model in the next Subsection 1.1, we present our main results in Section 2 that is composed of two parts: the asymptotics of the distribution of N in Subsection 2.1 and the asymptotics of the distribution of T in Subsection 2.2. In Subsection 2.1 we study three types of distinct behavior, i.e., the very heavy asymptotics in Subsection 2.1.1, the medium heavy (Weibull) asymptotics in Subsection 2.1.2 and the nearly exponential asymptotics in Subsection 2.1.3. Then, we concludes the paper with engineering implications in Section 3, which is followed by Section 4 that contains some of the technical proofs that have been deferred from the preceding sections.

1.1 Description of the Channel

In this section, we formally describe our model and provide necessary definitions and notation. Consider transmitting a generic data unit of random size L over a channel with failures. Without loss of generality, we assume that the channel is of unit capacity. The channel dynamics is modeled as an on-off process $\{(A_i, U_i)\}_{i\geqslant 1}$ with alternating independent periods when channel is available A_i and unavailable U_i, respectively. In each period of time that the channel becomes available, say A_i, we attempt to transmit the data unit and, if $L \leq A_i$, we say that the transmission was successful; otherwise, we wait for the next period A_{i+1} when the channel is available and attempt to retransmit the data from the beginning. A sketch of the model depicting the system is drawn in Figure 1.

Assume that $\{U, U_i\}_{i\geqslant 1}$ and $\{A, A_i\}_{i\geqslant 1}$ are two mutually independent sequences of i.i.d. random variables.

Definition 1.1 *The total number of (re)transmissions for a generic data unit of length L is defined as*

$$N \triangleq \inf\{n : A_n \geqslant L\},$$

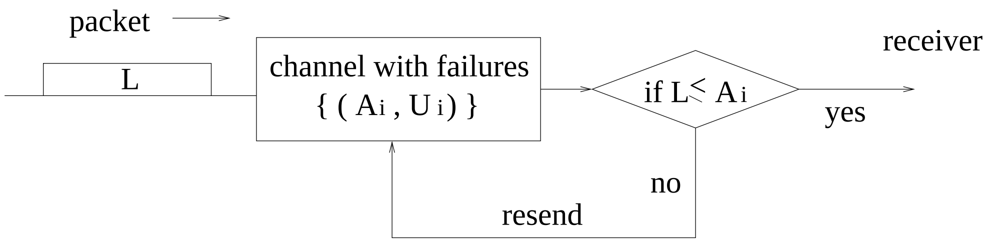

Figure 1: Packets sent over a channel with failures

and, the total transmission time for the data unit is defined as

$$T \triangleq \sum_{i=1}^{N-1} (A_i + U_i) + L. \tag{1.1}$$

We denote the complementary cumulative distribution functions for A and L, respectively, as

$$\bar{G}(x) \triangleq \mathbb{P}[A > x]$$

and

$$\bar{F}(x) \triangleq \mathbb{P}[L > x].$$

It was first discovered in Theorem 6 of [**?**] that this model leads to subexponential delay T under quite general conditions. The following slightly more general proposition was proven in Lemma 1 of [**?**] using probabilistic arguments (see also Proposition 1.2 in [**?**]).

Proposition 1.1 *If $\bar{F}(x) > 0$ for all $x \geqslant 0$, then both N and T are* subexponential *in the following sense that, for any $\epsilon > 0$,*

$$e^{\epsilon n}\mathbb{P}[N > n] \to \infty \quad as\, n \to \infty \tag{1.2}$$

and

$$e^{\epsilon t}\mathbb{P}[T > t] \to \infty \quad as\, t \to \infty. \tag{1.3}$$

Clearly, the preceding proposition defines a class of subexponential distributions that are induced by retransmissions; the **proof** of this proposition is presented in Subsection 4.1 for readers' convenience. The main study of this paper is to uncover the detailed structure of this class of distributions. More precisely, we investigate how the functional dependence of $\bar{F}$ and $\bar{G}$ (stated in the form $\bar{F}^{-1}(x) \approx \Phi(\bar{G}^{-1}(x))$) impacts the tail behavior of the distributions of both N and T, and the exact meaning of $\approx$ will be defined according to the context.

2 Main Results

This section presents our main results. Here, we assume that $\bar{F}(x)$ is a continuous function with support on $[0, \infty)$. If $\bar{F}(x)$ is lattice valued, our results may still hold; see Remarks 3 and 6. If $\bar{F}(x)$ has only a finite support, we discuss this situation in Section 3; see also Example 3 in Section IV of [**?**] and Section 3 of [**?**]. According to (1.1), the total transmission time T naturally depends on the number of transmissions N, and therefore, we first study the distributional properties of the number of transmissions N in Subsection 2.1, and then evaluate the total transmission time T using the large deviation approach in Subsection 2.2.

2.1 Asymptotics of the Distribution of the Number of Retransmissions N

This subsection presents the asymptotic results for the number of retransmissions N depending on the functional relationship $\Phi(\cdot)$ between $\bar{F}$ and $\bar{G}$. Informally, we study three scenarios: very heavy asymptotics (when $\log(\Phi(n))$ is slowly varying), medium heavy (Weibull) asymptotics (when $\log(\Phi(n))$ is regularly varying), and nearly exponential (when $\log\log(\Phi(n))$ is regularly varying), where within and between these subclasses we also identify critical functional points that define different distributional behavior of N.

More precisely, we show that:

1. If $\Phi(n)$ is dominantly regularly varying, e.g., regularly varying, then $\mathbb{P}[N > n] \approx \Phi(n)^{-1}$, as stated in Proposition 2.2 and Theorem 2.1.

2. If $\Phi(n)$ is not dominantly regularly varying, e.g., $\Phi(n)^{-1}$ being lognormal, the preceding tail equivalence $\mathbb{P}[N > n] \approx \Phi(n)^{-1}$ basically does not hold, as shown in Proposition 2.3. However, we show in a weaker form that, if $\log(\Phi(n))$ is slowly varying, then $\log(\mathbb{P}[N > n])$ is essentially slowly varying as well, as proved in Proposition 2.1. Interestingly, within this class, we discover two types of distinct functional behavior of $\log \mathbb{P}[N > n]$ depending on the growth of $\log(\Phi(n))$:

   (a) If $\log(\Phi(n))$ grows slower than $e^{\sqrt{\log n}}$, then we have the asymptotic equivalence $\log(\mathbb{P}[N > n]) \approx -\log(\Phi(n))$, as shown in Theorem 2.2.

   (b) This asymptotic equivalence does not hold if $\log(\Phi(n))$ grows faster than $e^{\sqrt{\log n}}$, and we demonstrate a different functional form in Proposition 2.5.

3. If $\log(\Phi(n))$ is regularly varying with index $\beta > 0$, then basically one obtains a Weibull distribution for N, $\log \mathbb{P}[N > n] \approx -(\log \Phi(n))^{1/(\beta+1)}$, as presented in Theorem 2.3.

4. When the decay of $\mathbb{P}[L > x]$ is much faster than $\mathbb{P}[A > x]$, i.e., $\log\log \mathbb{P}[L > x]^{-1} \approx R_\gamma(\log \mathbb{P}[A > x]^{-1})$ with $R_\gamma(\cdot), \gamma > 1$ being regularly varying, we obtain nearly exponential distributions for N in the form $\log(\mathbb{P}[N > n]) \approx n/R_\gamma^{\leftarrow}(\log n)$ with $R_\gamma^{\leftarrow}(n)$ being regularly varying with $0 < 1/\gamma < 1$, implying that $R_\gamma^{\leftarrow}(\log n)$ is slowly varying; see Theorem 2.4.

Our **proving method** is based on the following two key arguments:

1. $\Phi(x)$ is eventually monotonically increasing, which guarantees the existence of an inverse function $\Phi^{\leftarrow}(x)$ of $\Phi(x)$ when x is large enough.

2. $\bar{F}(x)$ is continuous, which implies that $V \triangleq \bar{F}(L)$ is a uniform random variable on $(0, 1)$, e.g., see Proposition 2.1 in Chapter 10 of [**?**]. Furthermore, our method essentially extends to lattice valued $\bar{F}(x)$ as well, as discussed in Remarks 3 and 6.

2.1.1 Very Heavy Asymptotics

This subsection studies the situation when the distribution of the number of retransmissions N is heavier than Weibull distributions. Specifically, we answer under what conditions $\mathbb{P}[N > n] \approx \Phi(n)^{-1}$ holds assuming $\bar{F}^{-1}(x) \approx \Phi(\bar{G}^{-1}(x))$, meaning that the complementary cumulative distribution function of N is of the same form (in terms of $\Phi(\cdot)$) as the functional relationship $\Phi(\cdot)$ between $\bar{F}$ and $\bar{G}$.

We term this subclass very heavy distributions since if $\log(\Phi(\cdot))$ is slowly varying, then the number of retransmissions N is always heavier than Weibull distribution, which is stated in the following Proposition 2.1.

Proposition 2.1 *If* $\log(\Phi(\cdot))$ *is slowly varying and*

$$\lim_{x\to\infty} \frac{\log\left(\bar{F}(x)^{-1}\right)}{\log\left(\Phi\left(\bar{G}(x)^{-1}\right)\right)} = 1, \tag{2.1}$$

then, for any $\epsilon > 0$*, as* $n \to \infty$*,*

$$\lim_{n\to\infty} \frac{\log\left(\mathbb{P}[N > n]^{-1}\right)}{n^{\epsilon}} = 0.$$

The **proof** of this proposition will be presented in Subsection 4.2. In the remainder of this subsection we study the detailed structure of this class of distributions that have very heavy tails. The Weibull distribution will be studied in the next Subsection 2.1.2 on medium heavy asymptotics.

Definition 2.1 *For an eventually non-decreasing function* $\Phi(x) : \mathbb{R}^+ \to \mathbb{R}^+$*, we say that* $\Phi(x)$ *is dominantly regularly varying if*

$$\overline{\lim_{x\to\infty}} \frac{\Phi(ex)}{\Phi(x)} < \infty, \tag{2.2}$$

where $e \equiv \exp(1)$*.*

In the paper we use the following standard notation. For any two real functions $a(t)$ and $b(t)$ and fixed $t_0 \in \mathbb{R} \cup \{\infty\}$, we use $a(t) \sim b(t)$ as $t \to t_0$ to denote $\lim_{t\to t_0}[a(t)/b(t)] = 1$. Similarly, we say that $a(t) \gtrsim b(t)$ as $t \to t_0$ if $\underline{\lim}_{t\to t_0} a(t)/b(t) \geq 1$; $a(t) \lesssim b(t)$ has a complementary definition. In addition, we say that $a(t) = o(b(t))$ as $t \to t_0$ if $\lim_{t\to t_0} a(t)/b(t) = 0$. When $t_0 = \infty$, we often simply write $a(t) = o(b(t))$ without explicitly stating $t \to \infty$ in order to simplify the notation. Also, we use the standard definition of an inverse function $f^{\leftarrow}(x) \triangleq \inf\{y : f(y) > x\}$ for a non-decreasing function $f(x)$; note that the notation $f^{-1}(x)$ is reserved for $1/f(x)$.

The following two propositions show that $\mathbb{P}[N > n]$ is tail equivalent to $\Phi(n)^{-1}$ basically only when $\Phi(n)$ is dominantly regularly varying.

Proposition 2.2 *If, as* $x \to \infty$*,*

$$\bar{F}^{-1}(x) \sim \Phi(\bar{G}^{-1}(x)), \tag{2.3}$$

then, there is finite $c \geqslant 1$ *such that*

$$c^{-1} \leqslant \underline{\lim_{n\to\infty}} \mathbb{P}[N > n]\Phi(n) \leqslant \overline{\lim_{n\to\infty}} \mathbb{P}[N > n]\Phi(n) \leqslant c.$$

Remark 1 Note that for this result as well as those in the rest of the paper we could have equivalently assumed that $\bar{F}(x) \sim \Phi(\bar{G}(x))$ where $\Phi(\cdot)$ is eventually non-increasing and satisfies the appropriate regularity conditions in the neighborhood of 0, e.g., condition (2.3) would be restated in the neighborhood of 0. In this case, the respective statement would be in the form $\mathbb{P}[N > n] \approx \Phi(n^{-1})$. Furthermore, the current form has additional notational benefits in the later sections, e.g., $\log\log\Phi(n)$ would need to be replaced by $\log\left(-\log(\Phi(n^{-1}))\right)$ in (say) Proposition 2.5.

Proposition 2.3 *If (2.3) is satisfied and* $\Phi(x)$ *is eventually non-decreasing with*

$$\lim_{x\to\infty} \frac{\Phi(ex)}{\Phi(x)} = \infty,$$

then,

$$\lim_{n\to\infty} \mathbb{P}[N > n]\Phi(n) = \infty.$$

When $\Phi(\cdot)$ is regularly varying, which is a subset of the dominantly regularly varying functions, we can compute the exact asymptotics of the distribution of N.

Theorem 2.1 *Assuming $\bar{F}^{-1}(x) \sim \Phi\left(\bar{G}^{-1}(x)\right)$ where $\Phi(\cdot)$ is regularly varying with index α, we obtain:*

i) If $\alpha > 0$, then, as $n \to \infty$,

$$\mathbb{P}[N > n] \sim \frac{\Gamma(\alpha+1)}{\Phi\left(n\right)}. \tag{2.4}$$

ii) If $\alpha = 0$ (meaning $\Phi(\cdot)$ is slowly varying) and $\Phi(x)$ is eventually non-decreasing, then, as $n \to \infty$,

$$\mathbb{P}[N > n] \sim \frac{1}{\Phi(n)}. \tag{2.5}$$

Remark 2 For $\alpha > 0$, this theorem was proved in Theorem 4 of [**?**] using the method that we further expand in this paper; alternatively, a similar result for T was proved using Tauberian method in Theorem 2.2 of [**?**]. We will prove the corresponding result for T in Theorem 2.5 in Subsection 2.2.

Remark 3 (Lattice variables) Note that if $\bar{F}(x)$ and $\bar{G}(x)$ are *lattice valued*, then the distribution of N may still be tail equivalent to $\Phi(n)^{-1}$, as in Proposition 2.3, but the constant in front of $\Phi(n)^{-1}$ may be different from $\Gamma(\alpha+1)$, e.g., if $\mathbb{P}[L > n] \sim e^{-pn}, p > 0$ and $\mathbb{P}[A > n] \sim e^{-qn}, q > 0$, then this constant is between $e^{-p}\Gamma(1 + p/q)$ and $e^{p}\Gamma(1 + p/)$.

Before moving to the proof, we state two straightforward consequences of the preceding theorems; see also Theorem 1 and Corollary 1 in [**?**]. The following corollary allows $\bar{F}$ and $\bar{G}$ to have exponential type distributions, and the corresponding result for T was first derived in Theorem 7 of [**?**].

Corollary 2.1 *Assume that $\bar{G}(x) \sim e^{-\beta x}$ and $\bar{F}(x) \sim a x^{b} e^{-\delta x}$ where $b \in \mathbb{R}$ and $a, \beta > 0$, then,*

$$\mathbb{P}[N > n] \sim a\Gamma\left(\frac{\delta}{\beta} + 1\right)\beta^{-b}\frac{(\log t)^{b}}{t^{\frac{\delta}{\beta}}}. \tag{2.6}$$

Proof: It is easy to verify that, as $x \to \infty$,

$$\bar{F}^{-1}(x) \sim a^{-1}\beta^{b}\left(\log \bar{G}^{-1}(x)\right)^{-b}\bar{G}(x)^{-\frac{\delta}{\beta}},$$

and, therefore, we can choose

$$\Phi(x) = a^{-1}\beta^{b}\left(\log x\right)^{-b}x^{\frac{\delta}{\beta}},$$

which, by using Theorem 2.5, finishes the proof. □

The following corollary allows $\bar{F}$ and $\bar{G}$ to have normal-like distributions, i.e., much lighter tails than exponential distributions, as shown in Corollary 1 of [**?**] (see also Corollary 2.2 in [**?**]).

Corollary 2.2 *Suppose $\bar{G}(x) = \mathbb{P}[|N(0,\sigma_A^2)| > x]$ and $\bar{F}(x) = \mathbb{P}[|N(0,\sigma_L^2)| > x]$, where $N(0,\sigma^2)$ is a Gaussian random variable with mean zero and variance σ^2, then,*

$$\mathbb{P}[N > n] \sim \Gamma\left(\alpha+1\right)\alpha^{-1/2}\frac{(\pi \log n)^{\frac{1}{2}(\alpha-1)}}{n^{\alpha}}, \tag{2.7}$$

where $\alpha = {\sigma_A}^2/{\sigma_L}^2$.

Proof: First, notice that

$$\mathbb{P}[|N(0,\sigma^2)| > x] \sim \frac{2\sigma}{\sqrt{2\pi}x} e^{-\frac{x^2}{2\sigma^2}},$$

and therefore, recalling $\alpha = {\sigma_A}^2/{\sigma_L}^2$, we obtain

$$\bar{F}(x) \sim \pi^{\frac{1}{2}(\alpha-1)}\alpha^{-1/2}\left(-\log \bar{G}(x)\right)^{\frac{1}{2}(\alpha-1)}\left(\bar{G}(x)\right)^{\alpha}.$$

Hence, $\bar{F}(x)$ and $\bar{G}(x)$ satisfy the assumption of Theorem 2.1 with

$$\Phi(x) = \alpha^{1/2}(\pi \log x)^{\frac{1}{2}(1-\alpha)} x^{\alpha},$$

which implies (2.7). □

Next, we present the proofs for the preceding Propositions 2.2, 2.3 and Theorem 2.1. Note that the following proof represents a basis for the other proofs in this paper.

Proof: [of Proposition 2.2] Notice that the number of retransmissions is geometrically distributed given the packet size L,

$$\mathbb{P}[N > n \mid L] = (1 - \bar{G}(L))^n$$

and, therefore,

$$\mathbb{P}[N > n] = \mathbb{E}[(1 - \bar{G}(L))^n]. \tag{2.8}$$

Since $\Phi(x)$ is eventually non-decreasing, there exists x_0 such that for all $x > x_0$, $\Phi(x)$ has an inverse function $\Phi^{\leftarrow}(x)$. The condition (2.3) implies that, for $0 < \epsilon < 1$, there exists x_ϵ, such that for $x > x_\epsilon$,

$$(1-\epsilon)\bar{F}^{-1}(x) \leqslant \Phi(\bar{G}^{-1}(x)) \leqslant (1+\epsilon)\bar{F}^{-1}(x),$$

and thus, by choosing $x_\epsilon > x_0$, we obtain

$$\Phi^{\leftarrow}\left((1-\epsilon)\bar{F}^{-1}(x)\right) \leqslant \bar{G}^{-1}(x) \leqslant \Phi^{\leftarrow}\left((1+\epsilon)\bar{F}^{-1}(x)\right). \tag{2.9}$$

First, we will prove the *upper bound*. Recalling (2.8), noting that $V \triangleq \bar{F}(L)$ is a uniform random variable on $(0,1)$ (e.g., see Proposition 2.1 in Chapter 10 of [?]) and using (2.9), we obtain, for large n,

$$\begin{aligned}
\mathbb{P}[N > n] &= \mathbb{E}[(1-\bar{G}(L))^n] \\
&= \mathbb{E}[(1-\bar{G}(L))^n \mathbf{1}(L > x_\epsilon)] + \mathbb{E}[\left(1-\bar{G}(L)\right)^n \mathbf{1}(L \leqslant x_\epsilon)] \\
&\leqslant \mathbb{E}\left[e^{-\frac{n}{\Phi^{\leftarrow}((1+\epsilon)V^{-1})}}\right] + \left(1-\bar{G}(x_\epsilon)\right)^n \\
&\leqslant \mathbb{P}\left[0 \leqslant \frac{n}{\Phi^{\leftarrow}((1+\epsilon)V^{-1})} \leqslant 1\right] + \sum_{k=0}^{\lceil \log(\epsilon n)\rceil} e^{-e^k}\mathbb{P}\left[e^k \leqslant \frac{n}{\Phi^{\leftarrow}((1+\epsilon)V^{-1})} \leqslant e^{k+1}\right] \\
&\quad + e^{-e^{\lceil \log(\epsilon n)\rceil+1}}\mathbb{P}\left[\frac{n}{\Phi^{\leftarrow}((1+\epsilon)V^{-1})} > e^{\lceil \log(\epsilon n)\rceil+1}\right] + \left(1-\bar{G}(x_\epsilon)\right)^n \\
&\leqslant \frac{1+\epsilon}{\Phi(n)} + \sum_{k=0}^{\lceil \log(\epsilon n)\rceil} e^{-e^k}\frac{1+\epsilon}{\Phi\left(\frac{n}{e^{k+1}}\right)} + e^{-\epsilon n} + \left(1-\bar{G}(x_\epsilon)\right)^n.
\end{aligned} \tag{2.10}$$

The condition (2.2) implies that there exist finite n_d and d, such that for $n > n_d$,

$$\frac{\Phi(n)}{\Phi(n/e)} < d,$$

resulting in, for all k satisfying $n/e^k > n_d$,

$$\frac{\Phi\left(n\right)}{\Phi\left(\frac{n}{e^{k+1}}\right)} \leqslant d^{k+2}, \tag{2.11}$$

and therefore,

$$\Phi(n) \leqslant \Phi(n_d) d^{\log\left(\frac{n}{n_d}\right)+1},$$

which, in conjunction with (2.10), yields

$$\begin{aligned}\varlimsup_{n\to\infty} \mathbb{P}[N > n]\Phi(n) &\leqslant 1 + \epsilon + \sum_{k=0}^{\infty}(1+\epsilon)e^{-e^k} d^{k+2} \\ &\quad + \varlimsup_{n\to\infty} \left(e^{-\epsilon n} + \left(1 - \bar{G}(x_\epsilon)\right)^n\right) \Phi(n_d) d^{\log\left(\frac{n}{n_d}\right)+1} \\ &= 1 + \epsilon + \sum_{k=0}^{\infty}(1+\epsilon)e^{-e^k} d^{k+2} < \infty. \end{aligned} \tag{2.12}$$

Next, we prove the *lower bound*. Recalling (2.9) and choose $n > x_0$, we obtain

$$\begin{aligned}\mathbb{P}[N > n] &= \mathbb{E}[(1 - \bar{G}(L))^n] \\ &\geqslant \left(1 - \frac{1}{n}\right)^n \mathbb{P}\left[\bar{G}(L) \leqslant \frac{1}{n}\right] \\ &\geqslant \left(1 - \frac{1}{n}\right)^n \mathbb{P}\left[\Phi^{\leftarrow}\left((1-\epsilon)\bar{F}^{-1}(L)\right) \geqslant n\right] \\ &\geqslant \left(1 - \frac{1}{n}\right)^n \frac{1-\epsilon}{\Phi(n)}\,, \end{aligned}$$

implying

$$\varliminf_{n\to\infty} \mathbb{P}[N > n]\Phi(n) \geqslant \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n (1-\epsilon) = e^{-1}(1-\epsilon),$$

which, in conjunction with (2.12), proves the proposition. □

Proof: [of Proposition 2.3] Recalling (2.9) and choosing n large enough such that $\{\bar{G}(L) \leqslant e/n\} \subseteq \{L > x_\epsilon\}$ with x_ϵ being the same as chosen in (2.9), we obtain

$$\begin{aligned}\mathbb{P}[N > n] &= \mathbb{E}[(1 - \bar{G}(L))^n] \\ &\geqslant \left(1 - \frac{e}{n}\right)^n \mathbb{P}\left[\bar{G}(L) \leqslant \frac{e}{n}\right] \\ &\geqslant \left(1 - \frac{e}{n}\right)^n \mathbb{P}\left[\Phi^{\leftarrow}\left((1-\epsilon)\bar{F}^{-1}(L)\right) \geqslant \frac{n}{e}\right] \\ &\geqslant \left(1 - \frac{e}{n}\right)^n \frac{1-\epsilon}{\Phi\left(\frac{n}{e}\right)}\,, \end{aligned}$$

implying

$$\lim_{n\to\infty} \mathbb{P}[N>n]\Phi(n) \geqslant \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n \frac{(1-\epsilon)\Phi(n)}{\Phi\left(\frac{n}{e}\right)} = \infty,$$

which completes the proof. □

Proof: [of Theorem 2.1] We begin with proving (2.4). Without loss of generality, we can assume that $\Phi(x)$ is absolutely continuous and strictly monotone since, by Proposition 1.5.8 of [**?**], one can always find an absolutely continuous and strictly monotone function

$$\Phi^*(x) = \alpha \int_1^x \Phi(s)s^{-1}ds,\ x \geqslant 1, \tag{2.13}$$

which satisfies

$$\bar{F}^{-1}(x) \sim \Phi\left(\bar{G}^{-1}(x)\right) \sim \Phi^*\left(\bar{G}^{-1}(x)\right).$$

First, we prove the *upper bound*. Recalling (2.8), noting that $V \triangleq \bar{F}(L)$ is a uniform random variable on $(0,1)$ and using (2.9), we obtain

$$\begin{aligned}
\mathbb{P}[N>n] &= \mathbb{E}[(1-\bar{G}(L))^n] \\
&= \mathbb{E}[(1-\bar{G}(L))^n \mathbf{1}(L \geqslant x_\epsilon)] + \mathbb{E}[\left(1-\bar{G}(L)\right)^n \mathbf{1}(L < x_\epsilon)] \\
&\leqslant \mathbb{E}\left[e^{-\frac{n}{\Phi^{(-1)}((1+\epsilon)V^{-1})}}\right] + \left(1-\bar{G}(x_\epsilon)\right)^n.
\end{aligned} \tag{2.14}$$

Then, by choosing integers m and n_d (as in (2.11)) and noting that $\Phi(n)$ is regularly varying, the preceding inequality yields, for large n,

$$\begin{aligned}
\mathbb{P}[N>n] &\leqslant \mathbb{E}\left[e^{-\frac{n}{\Phi^{(-1)}((1+\epsilon)V^{-1})}} \mathbf{1}\left(0 \leqslant \frac{n}{\Phi^{\leftarrow}((1+\epsilon)V^{-1})} \leqslant e^m\right)\right] \\
&\quad + \sum_{k=m}^{\log\left(\frac{n}{n_d}\right)-1} e^{-e^k} \mathbb{P}\left[e^k \leqslant \frac{n}{\Phi^{\leftarrow}((1+\epsilon)V^{-1})} \leqslant e^{k+1}\right] + o\left(\frac{1}{\Phi(n)}\right) \\
&\leqslant \int_0^{e^m} e^{-z}\left(\frac{\Phi'(n/z)}{\Phi^2(n/z)}\frac{(1+\epsilon)n}{z^2}\right)dz + \sum_{k=m}^{\log\left(\frac{n}{n_d}\right)-1} e^{-e^k}\frac{1+\epsilon}{\Phi\left(\frac{n}{e^{k+1}}\right)} + o\left(\frac{1}{\Phi(n)}\right),
\end{aligned}$$

resulting in

$$\begin{aligned}
\mathbb{P}[N>n]\Phi(n) &\leqslant \int_0^{e^m} \frac{\Phi(n)}{\Phi(n/z)}\frac{\Phi'(n/z)}{\Phi(n/z)}\frac{e^{-z}(1+\epsilon)n}{z^2}dz \\
&\quad + \sum_{k=m}^{\log\left(\frac{n}{n_d}\right)-1} (1+\epsilon)e^{-e^k}\frac{\Phi(n)}{\Phi\left(\frac{n}{e^{k+1}}\right)} + \Phi(n)\left(1-\bar{G}(x_\epsilon)\right)^n \\
&\triangleq I_1 + I_2 + I_3.
\end{aligned} \tag{2.15}$$

Since regularly varying functions are also dominantly regularly varying, the bound in (2.11) implies

$$I_2 \leqslant \sum_{k=m}^{\log\left(\frac{n}{n_d}\right)-1} (1+\epsilon)e^{-e^k}d^{k+2} \leqslant \sum_{k=m}^{\infty}(1+\epsilon)e^{-e^k}d^{k+2} < \infty. \tag{2.16}$$

For I_1, since $\Phi(n)$ is regularly varying, by the Characterisation Theorem of regular variation (e.g., see Theorem 1.4.1 of [?]) and the uniform convergence theorem of slowly varying functions (Theorem 1.2.1 of [?]), it is easy to obtain uniformly for $0 \leqslant z \leqslant e^m$, as $n \to \infty$,

$$\frac{\Phi(n)}{\Phi(n/z)} \sim z^\alpha$$

and, recalling (2.13),

$$\frac{\Phi'(n/z)}{\Phi(n/z)} = \frac{z\alpha}{n},$$

which implies

$$I_1 \sim \int_0^{e^m} (1+\epsilon)\alpha e^{-z} z^{\alpha-1} dz. \tag{2.17}$$

Furthermore, $\Phi(n)$ being regularly varying implies that $I_3 \to 0$ as $n \to \infty$. Thus, passing $n \to \infty$ in (2.15), recalling (2.16) and then passing $m \to \infty$, $\epsilon \to 0$, we obtain

$$\mathbb{P}[N > n]\Phi(n) \lesssim \int_0^{\infty} \alpha e^{-z} z^{\alpha-1} dz = \Gamma(\alpha+1). \tag{2.18}$$

As for the *lower bound*, the proof follows similar arguments, and the details are presented in Subsection 4.3. The same subsection also contains the proof of the statement ii) of the theorem. □

The condition of $\Phi(\cdot)$ being dominantly varying is basically necessary in order for $\mathbb{P}[N > n] \approx \Phi(n)^{-1}$ to hold. As shown in Proposition 2.4, this tail equivalence basically does not hold if $\Phi(\cdot)$ is not dominantly varying, e.g., if $\Phi(\cdot)^{-1}$ is lognormal. Here, we further characterize the behavior of the lognormal type distributions in the following proposition.

Proposition 2.4 *If* $\log(\Phi(x)) = \lambda(\log x)^\delta$, $\delta > 1, \lambda > 0$, *then, under the condition (2.3), we obtain*

$$\lim_{n\to\infty} \frac{\log\left(\mathbb{P}[N > n]^{-1}\right) - \log(\Phi(x))}{(\log\log n)(\log n)^{\delta-1}} = -\lambda\delta(\delta-1).$$

The **proof** of this proposition is presented in Subsection 4.4.

Remark 4 In Proposition 2.4, it can be easily verified that $\Phi(\cdot)$ is not dominantly regularly varying, and therefore, according to Propositions 2.2 and 2.3, we know $\mathbb{P}[N > n]\Phi(x) \to \infty$ as $n \to \infty$. However, Proposition 2.4 further characterize how fast $\mathbb{P}[N > n]\Phi(n)$ goes to infinity in the logarithmic scale, which also implies a weaker result

$$\lim_{n\to\infty} \frac{\log\left(\mathbb{P}[N > n]^{-1}\right)}{\log(\Phi(n))} = 1.$$

In the following theorem we extend the preceding logarithmic limit under a more general condition on $\Phi(\cdot)$.

Theorem 2.2 *If an eventually non-decreasing function* $\Phi(x) \triangleq e^{l(x)}$ *satisfies (2.1) where* $l(x)$ *is slowly varying with*

$$\lim_{x\to\infty} \frac{l\left(\frac{x}{l(x)}\right)}{l(x)} = 1, \tag{2.19}$$

then,

$$\lim_{n\to\infty} \frac{\log\left(\mathbb{P}[N > n]^{-1}\right)}{\log \Phi(n)} = 1. \tag{2.20}$$

Remark 5 Note that if $\log(\Phi(x)) = e^{(\log x)^\delta}$ then the condition (2.19) is satisfied if $0 < \delta < 1/2$ and it does not hold if $\delta \geqslant 1/2$, which can be easily verified. Furthermore, if $\log(\Phi(x)) = \Psi(\log x)$ where $\Psi(x)$ is regularly varying, e.g., $\Phi(x)^{-1}$ being lognormal, then the condition (2.19) also holds, which is stated in the following corollary.

Remark 6 (Lattice variables) When L is lattice valued, it is easy to see from the proof of Theorem 2.2 that, if there exists a continuous random variable L^* such that $\log \mathbb{P}[L^* > x] \sim \log \mathbb{P}[L > x]$ as $x \to \infty$, or equivalently, if there exists a continuous negative non-increasing function $q(x)$ such that $\log \mathbb{P}[L > x] \sim q(x)$, then Theorem 2.2 still holds, e.g., when L has a geometric or Poisson distribution. To rigorously prove this claim, one can use similar arguments as in the proof of Theorem 3.1 in Section 3 of this paper. Note that this remark also applies to other logarithmic asymptotics, e.g., see Corollary 2.3, Propositions 2.5 and 2.6, and Theorems 2.3, 2.4, 2.6 and 2.7.

Corollary 2.3 *If a regularly varying function $\Psi(\cdot)$ with a non-negative index satisfies*

$$\lim_{x\to\infty} \frac{\log \bar{F}(x)^{-1}}{\Psi\left(\log \bar{G}(x)^{-1}\right)} = 1$$

and, in addition, is eventually non-decreasing when $\Psi(\cdot)$ is slowly varying, then, we have

$$\lim_{n\to\infty} \frac{\log\left(\mathbb{P}[N > n]^{-1}\right)}{\Psi(\log n)} = 1.$$

Remark 7 This result, or more precisely Theorem 2.6 in Subsection 2.2, implies parts (1:1), (2:1) and (2:2) of Theorem 2.1 in [**?**] and extends Theorem 2 in [**?**].

Proof: [of Corollary 2.3] For a regularly varying function $\Psi(\cdot)$, it is easy to verify that $l(x) = \Psi(\log(x))$ satisfies

$$\lim_{x\to\infty} \frac{l\left(\frac{x}{l(x)}\right)}{l(x)} = \lim_{x\to\infty} \frac{\Psi\left(\log x - \log \Psi(\log(x))\right)}{\Psi(\log(x))} = 1,$$

and therefore, by Theorem 2.2, we prove the corollary. □

Remark 8 Note that, in conjunction with Remark 5, the condition (2.19) is close to necessary since the result (2.20) does not hold if $\log(\Phi(x)) = e^{(\log x)^\delta}$, $1/2 < \delta < 1$, as can be seen from the following proposition.

Proposition 2.5 *If* $\log(\Phi(x)) = e^{\lambda(\log x)^\delta}$, $1/2 < \delta < 1$, $\lambda > 0$, *then, under the condition (2.1), we obtain*

$$\log\left(\log\left(\mathbb{P}[N > n]^{-1}\right)\right) - \log\left(\log(\Phi(x))\right) \sim -\delta\lambda^2(\log n)^{2\delta-1}.$$

The **proof** of this proposition is presented in Subsection 4.5.

Remark 9 Note that this result implies that, for $0 < \epsilon < 1$ and n large,

$$0 \leqslant \frac{\log\left(\mathbb{P}[N > n]^{-1}\right)}{\log \Phi(n)} \leqslant e^{-(1-\epsilon)\alpha\lambda^2(\log n)^{2\alpha-1}} \to 0,$$

which contrasts the limit in (2.20).

Proof: [of Theorem 2.2] Since $\Phi(x)$ is eventually non-decreasing, there exists x_0 such that for all $x > x_0$, $\Phi(x)$ has an inverse function $\Phi^{\leftarrow}(x)$. The condition (2.1) implies that, for $0 < \epsilon < 1$, there exists x_ϵ, such that for $x > x_\epsilon$,

$$\bar{F}^{-(1-\epsilon)}(x) \leqslant \Phi(\bar{G}^{-1}(x)) \leqslant \bar{F}^{-(1+\epsilon)}(x),$$

thus, choosing $x_\epsilon > x_0$, we obtain

$$\Phi^{\leftarrow}\left(\bar{F}^{-(1-\epsilon)}(x)\right) \leqslant \bar{G}^{-1}(x) \leqslant \Phi^{\leftarrow}\left(\bar{F}^{-(1+\epsilon)}(x)\right). \tag{2.21}$$

First, we prove the *upper bound*. Recalling (2.8), noting that $V \triangleq \bar{F}(L)$ is a uniform random variable on $(0,1)$, and using (2.21), we obtain, for integer y and large n,

$$\begin{aligned}
\mathbb{P}[N > n] &= \mathbb{E}[(1-\bar{G}(L))^n] \\
&= \mathbb{E}[(1-\bar{G}(L))^n \mathbf{1}(L > x_\epsilon)] + \mathbb{E}[\big(1-\bar{G}(L)\big)^n \mathbf{1}(L \leqslant x_\epsilon)] \\
&\leqslant \mathbb{E}\left[e^{-\frac{n}{\Phi^{\leftarrow}\left(V^{-(1+\epsilon)}\right)}}\right] + \big(1-\bar{G}(x_\epsilon)\big)^n \\
&\leqslant \sum_{k=0}^{y} e^{-k}\mathbb{P}\left[k \leqslant \frac{n}{\Phi^{\leftarrow}\left(V^{-(1+\epsilon)}\right)} \leqslant k+1\right] + e^{-(y+1)} + \big(1-\bar{G}(x_\epsilon)\big)^n,
\end{aligned}$$

which, by Proposition 1.1, noting $\Phi(x) = e^{l(x)}$ and choosing $y = \lceil l(n) \rceil - 1$, implies

$$\begin{aligned}
\mathbb{P}[N > n] &\leqslant \sum_{k=0}^{\lceil l(n)\rceil - 1} e^{-k-\frac{1}{1+\epsilon} l\left(\frac{n}{k+1}\right)} + e^{-l(n)} + o\left(\mathbb{P}[N > n]\right) \\
&\leqslant \lceil l(n) \rceil e^{-\frac{1}{1+\epsilon} l\left(\frac{n}{\lceil l(n)\rceil}\right)} + e^{-l(n)} + o\left(\mathbb{P}[N > n]\right).
\end{aligned} \tag{2.22}$$

From (2.1), it is easy to see that $l(x)$ increases to infinity when $x \to \infty$ and, since $l(x)$ is slowly varying, by (2.19) and (2.22), we obtain

$$\varliminf_{n\to\infty} \frac{\log \mathbb{P}[N > n]^{-1}}{l(n)} \geqslant 1. \tag{2.23}$$

Next, we prove the *lower bound*. Recalling (2.21) and choosing n large enough, we obtain

$$\begin{aligned}
\mathbb{P}[N > n] &= \mathbb{E}[(1-\bar{G}(L))^n] \\
&\geqslant \left(1 - \frac{1}{n}\right)^n \mathbb{P}\left[\bar{G}(L) \leqslant \frac{1}{n}\right] \\
&\geqslant \left(1 - \frac{1}{n}\right)^n \mathbb{P}\left[\Phi^{\leftarrow}\left(\bar{F}^{-(1-\epsilon)}(L)\right) \geqslant n\right] \\
&\geqslant \left(1 - \frac{1}{n}\right)^n \frac{1}{\Phi(n)^{\frac{1}{1-\epsilon}}},
\end{aligned}$$

implying

$$\varlimsup_{n\to\infty} \frac{\log \mathbb{P}[N > n]^{-1}}{l(n)} \leqslant \frac{1}{1-\epsilon},$$

which, by passing $\epsilon \to 0$ and in conjunction with (2.23), proves the theorem. □

2.1.2 Medium Heavy (Weibull) Asymptotics

In the preceding subsection, we studied the scenario when the distribution of N is heavier than any Weibull distribution. Specifically, we establish the necessary conditions under which $\mathbb{P}[N > n] \approx \Phi^{-1}(n)$ holds when the separation between $\mathbb{P}[L > x]$ and $\mathbb{P}[A > x]$ can be characterized in the form of $\Phi(x) = e^{l(x)}$ with $l(x)$ being slowly varying. In this subsection, we further increase the separation in the sense that $\Phi(x) = e^{R_\beta(x)}$ with $R_\beta(x)$ being regularly varying of index $\beta > 0$, and under this condition the distribution of N is shown to be of Weibull type. In this situation, the tail equivalence developed in the preceding subsection does not hold anymore and admits a different form, as stated in the following theorem.

Theorem 2.3 *If an eventually non-decreasing function $\Phi(x) \triangleq e^{R_\beta(x)}$ satisfies (2.1) where $R_\beta(x) \equiv x^\beta l(x)$, $\beta > 0$ is regularly varying with $l(x)$ satisfying*

$$\lim_{x\to\infty} \frac{l\left(\left(\frac{x}{l(x)}\right)^{\frac{1}{1+\beta}}\right)}{l(x)} = 1, \tag{2.24}$$

then,

$$\lim_{n\to\infty} \frac{\log\left(\mathbb{P}[N > n]^{-1}\right)}{\left(\log \Phi(n)\right)^{\frac{1}{\beta+1}}} = \beta^{\frac{1}{\beta+1}} + \beta^{-\frac{\beta}{\beta+1}}. \tag{2.25}$$

Remark 10 This theorem, or more precisely Theorem 2.7 of the following Subsection 2.2, implies part (1:2) of Theorem 2.1 in [**?**], and provides a more precise logarithmic asymptotics instead of a double logarithmic limit that was proved in [**?**]. Furthermore, although the condition (2.24) appears complicated, it is easy to check that any slowly varying function $l(x) = l_1(\log x)$ satisfies it, where $l_1(\cdot)$ is also a slowly varying function.

Proof: [of Theorem 2.3] First, we begin with proving the *upper bound*. Following the same approach as in the proof of Theorem 2.2, we obtain, for $\epsilon > 0$, integer y and n large enough,

$$\begin{aligned} \mathbb{P}[N > n] &\leqslant \sum_{k=0}^{y-1} e^{-k} \mathbb{P}\left[k \leqslant \frac{n}{\Phi^{\leftarrow}(V^{-(1+\epsilon)})} \leqslant k+1\right] + e^{-y} + o\left(\mathbb{P}[N > n]\right) \\ &\leqslant \sum_{k=0}^{y-1} e^{-k - \frac{1}{1+\epsilon} R_\beta\left(\frac{n}{k+1}\right)} + e^{-y} + o\left(\mathbb{P}[N > n]\right). \end{aligned} \tag{2.26}$$

Using the same argument as in (2.13), we can find an absolutely continuous and strictly increasing function $R_\beta^*(u) \triangleq \beta \int_1^u R_\beta(s) s^{-1} ds, u \geqslant 1$ that is a modified version of $R_\beta(u)$. This newly constructed function $R_\beta^*(u)$ satisfies that, for $0 < \epsilon < 1$, there exists $y_\epsilon > 0$, such that $(1-\epsilon)R_\beta^*(u) < R_\beta(u) < (1+\epsilon)R_\beta^*(u)$ for $u > y_\epsilon$. Therefore, for $0 < x < n/y_\epsilon$,

$$x + \frac{1}{1+\epsilon} R_\beta\left(\frac{n}{x}\right) \geqslant x + \frac{1-\epsilon}{1+\epsilon} R_\beta^*\left(\frac{n}{x}\right),$$

and, for $u \geqslant 1$,

$$\left(R_\beta^*(u)\right)' = \beta u^{\beta-1} l(u). \tag{2.27}$$

Choosing $y = \lceil n/y_\epsilon \rceil$ in (2.26) and using the asymptotic equivalence relationship between $R_\beta(\cdot)$ and $R^*_\beta(\cdot)$, we obtain

$$\begin{aligned}\mathbb{P}[N > n] &\leqslant \sum_{k=0}^{\left\lceil \frac{n}{y_\epsilon} \right\rceil - 1} e^{-k - \frac{1}{(1+\epsilon)} R_\beta\left(\frac{n}{k+1}\right)} + e^{-\frac{n}{y_\epsilon}} + o\left(\mathbb{P}[N > n]\right) \\ &\leqslant \sum_{k=0}^{\left\lceil \frac{n}{y_\epsilon} \right\rceil - 1} e^{-k - \frac{1-\epsilon}{1+\epsilon} R^*_\beta\left(\frac{n}{k+1}\right)} + o\left(\mathbb{P}[N > n]\right). \end{aligned} \tag{2.28}$$

Next, let $f(x) = x + R^*_\beta\left(n/x\right)(1-\epsilon)/(1+\epsilon)$, and suppose that $f(x)$ reaches the maximum at x^* for $0 < x \leqslant n/y_\epsilon$. From (2.27), it is easy to check that

$$f'(x) = 1 - \frac{1-\epsilon}{1+\epsilon}\left(R^*_\beta\left(\frac{n}{x}\right)\right)' \frac{n}{x^2} = 1 - \frac{1-\epsilon}{1+\epsilon}\frac{\beta}{n}\frac{n^{\beta+1}}{x^{\beta+1}} l\left(\frac{n}{x}\right).$$

Then, define $g(u) \triangleq u^{\beta+1} l\left(u\right)$, and use the same argument as in constructing $R^*_\beta(\cdot)$, we can find an absolutely continuous and strictly increasing function $g^*(u) \triangleq \beta \int_1^u u^\beta l\left(u\right) ds, u \geqslant 1$, such that $(1-\epsilon)g(u) < g^*(u) < (1+\epsilon)g(u), u > u_\epsilon$ for $u_\epsilon > 0$. Therefore, for $0 < x < n/u_\epsilon$, we obtain,

$$1 - \frac{1}{1+\epsilon}\frac{\beta}{n} g^*\left(\frac{n}{x}\right) < f'(x) = 1 - \frac{1-\epsilon}{1+\epsilon}\frac{\beta}{n} g\left(\frac{n}{x}\right) < 1 - \frac{1-\epsilon}{(1+\epsilon)^2}\frac{\beta}{n} g^*\left(\frac{n}{x}\right),$$

where, as shown in the preceding inequalities, the lower and upper bound of $f'(x)$ are two monotonically increasing functions for $0 < x < n$.

Now, define

$$x_1 \triangleq \left(\frac{(1-\epsilon)^3}{(1+\epsilon)^2}\right)^{\frac{1}{\beta+1}} \beta^{\frac{1}{\beta+1}} n^{\frac{\beta}{\beta+1}} l\left(n\right)^{\frac{1}{\beta+1}}$$

and

$$x_2 \triangleq (1+\epsilon)^{\frac{1}{\beta+1}} \beta^{\frac{1}{\beta+1}} n^{\frac{\beta}{\beta+1}} l\left(n\right)^{\frac{1}{\beta+1}}.$$

It is easy to see that, by condition (2.24), for n large enough,

$$f'(x_1) \leqslant 1 - \frac{1-\epsilon}{(1+\epsilon)^2}\frac{\beta}{n} g^*\left(\frac{n}{x_1}\right) < 1 - \left(\frac{1-\epsilon}{1+\epsilon}\right)^2 \frac{\beta}{n} g\left(\frac{n}{x_1}\right) < 0,$$

and

$$f'(x_2) \geqslant 1 - \frac{1}{1+\epsilon}\frac{\beta}{n} g^*\left(\frac{n}{x_2}\right) > 1 - \frac{\beta}{n} g\left(\frac{n}{x_2}\right) > 0,$$

which implies that, there exist $n_\epsilon > 0$ such that for all $n > n_\epsilon$,

$$x_1 < x^* < x_2. \tag{2.29}$$

Therefore, using (2.28), (2.29) and recalling $R_\beta(u) < (1+\epsilon)R^*_\beta(u)$ yields

$$\begin{aligned}\mathbb{P}[N > n] &\leqslant \left\lceil \frac{n}{y_\epsilon} \right\rceil e^{1 - f(x^*)} + o\left(\mathbb{P}[N > n]\right) \\ &\leqslant \left\lceil \frac{n}{y_\epsilon} \right\rceil e^{1 - x_1 - \frac{1}{(1+\epsilon)^2} R_\beta\left(\frac{n}{x_2}\right)} + o\left(\mathbb{P}[N > n]\right),\end{aligned}$$

resulting in

$$\varliminf_{n\to\infty} \frac{\log\left(\mathbb{P}[N>n]^{-1}\right)}{n^{\frac{\beta}{\beta+1}} l(n)^{\frac{1}{\beta+1}}} \geqslant \left(\frac{(1-\epsilon)^3}{(1+\epsilon)^2}\right)^{\frac{1}{\beta+1}} \beta^{\frac{1}{\beta+1}} + (1+\epsilon)^{-\frac{\beta}{\beta+1}-2} \beta^{-\frac{\beta}{\beta+1}}.$$

Passing $\epsilon \to 0$ in the preceding inequality yields

$$\varliminf_{n\to\infty} \frac{\log\left(\mathbb{P}[N>n]^{-1}\right)}{n^{\frac{\beta}{\beta+1}} l(n)^{\frac{1}{\beta+1}}} \geqslant \beta^{\frac{1}{\beta+1}} + \beta^{-\frac{\beta}{\beta+1}}. \tag{2.30}$$

Now, we proceed with proving the *lower bound*. By recalling the condition (2.21) and using $1-x \geqslant e^{-(1+\epsilon)x}$ for x small enough, we obtain, for n large enough and $x_0 > 0$,

$$\begin{aligned}
\mathbb{P}[N>n] &\geqslant \mathbb{E}[(1-\bar{G}(L))^n \mathbf{1}(L \geqslant x_\epsilon)] \geqslant \mathbb{E}[e^{-(1+\epsilon)\bar{G}(L)n}\mathbf{1}(L \geqslant x_\epsilon)] \\
&\geqslant \mathbb{E}\left[e^{-\frac{(1+\epsilon)n}{\Phi^{\leftarrow}\left(V^{-(1-\epsilon)}\right)}}\mathbf{1}(V \leqslant \bar{F}(x_\epsilon))\right] \geqslant e^{-x_0}\mathbb{P}\left[\frac{(1+\epsilon)n}{\Phi^{\leftarrow}\left(V^{-(1-\epsilon)}\right)} \leqslant x_0, V \leqslant \bar{F}(x_\epsilon)\right] \\
&= e^{-x_0}\Phi\left(\frac{(1+\epsilon)n}{x_0}\right)^{-\frac{1}{1-\epsilon}} = e^{-x_0 - \frac{1}{1-\epsilon}R_\beta\left(\frac{(1+\epsilon)n}{x_0}\right)},
\end{aligned}$$

since $\{(1+\epsilon)n/\Phi^{\leftarrow}\left(V^{-(1-\epsilon)}\right) \leqslant x_0\}$ implies $\{V \leqslant \bar{F}(x_\epsilon)\}$ for all n large enough. Next, by choosing $x_0 = \beta^{\frac{1}{\beta+1}} n^{\frac{\beta}{\beta+1}} l\,(n)^{\frac{1}{\beta+1}}$, using the condition (2.24), and then passing $n \to \infty$ as well as $\epsilon \to 0$, yields,

$$\varlimsup_{n\to\infty} \frac{\log\left(\mathbb{P}[N>n]^{-1}\right)}{n^{\frac{\beta}{\beta+1}} l(n)^{\frac{1}{\beta+1}}} \leqslant \beta^{\frac{1}{\beta+1}} + \beta^{-\frac{\beta}{\beta+1}}. \tag{2.31}$$

Finally, combining (2.30) and (2.31) finishes the proof. □

2.1.3 Nearly Exponential Asymptotics

In the preceding subsection, the functional separation between $\mathbb{P}[L > x]$ and $\mathbb{P}[A > x]$ can be characterized in the form of $\Phi(x) = e^{R_\gamma(x)}$ with $R_\gamma(x)$ being regularly varying. In this subsection, we investigate the situation when the separation in terms of $\Phi(x)$ is even larger than $e^{R_\gamma(x)}$, which leads to the nearly exponential asymptotics for $\mathbb{P}[N > n]$ in the following proposition and Theorem 2.4.

Proposition 2.6 *If* $\log(\bar{F}^{-1}(x)) \sim e^{\left(\log(\bar{G}^{-1}(x))\right)^\delta}$, $\delta > 1$, *then,*

$$\log\left(\log\left(\mathbb{P}[N>n]^{-1}\right)\right) - \log n + (\log n)^{\frac{1}{\delta}} \sim \frac{1}{\delta}(\log n)^{\frac{2}{\delta}-1}.$$

Remark 11 Observe that $\delta = 2$ represents another critical point since $(\log n)^{2/\delta - 1}$ converges to 0 or ∞ if $\delta > 2$ or $1 < \delta < 2$, respectively. Furthermore, the result shows that $\mathbb{P}[N > n] \approx \exp\left(-n/e^{(\log n)^{1/\delta}}\right)$, which means that N is nearly exponential because $e^{(\log n)^{1/\delta}}$ is slowly varying for $\delta > 1$ (see p. 16 in [**?**]). In addition, informally speaking, we point out that the case $\delta = 1$ corresponds to the Weibull case already covered by Theorem 2.3 in Subsection 2.1.2, meaning that this proposition describes the change in functional behavior on the boundary between the Weibull case and the nearly exponential one.

Proof: First, we prove the *upper bound*. Following the same approach as in the proof of Theorem 2.2, we obtain, for $\epsilon > 0$,

$$\mathbb{P}[N > n] \leqslant \sum_{k=0}^{n-1} e^{-k-\frac{1}{1+\epsilon}e^{(\log n - \log(k+1))^{\delta}}} + o\left(\mathbb{P}[N > n]\right). \tag{2.32}$$

Suppose that $f(x) \triangleq x + \frac{1}{1+\epsilon}e^{(\log n - \log x)^{\delta}}$ reaches the minimum at x^{*}. It is easy to see that $f'(x) = 1 - e^{(\log n - \log x)^{\delta}}/((1+\epsilon)x)$ is an increasing function in x on $(0, n)$. For $0 < \epsilon < 1$ define

$$x_1 \triangleq \frac{n}{e^{\left(\log n - (1-\epsilon)(\log n)^{1/\delta}\right)^{1/\delta}}},$$

and for n large enough, we obtain

$$f'(x_1) = 1 - \frac{e^{-(1-\epsilon)(\log n)^{1/\delta}}}{(1+\epsilon)} e^{\left(\log n - (1-\epsilon)(\log n)^{1/\delta}\right)^{1/\delta}} \leqslant 1 - \frac{e^{\epsilon(\log n)^{1/\delta} - (1-\epsilon^2)(\log n)^{2/\delta - 1}/\delta}}{1+\epsilon} < 0,$$

implying $f(x)' < 0$ for $x < x_1$. Therefore, the minimum point x^{*} satisfies

$$x^{*} \geqslant x_1. \tag{2.33}$$

Combining (2.32) and (2.33), we obtain, for n large,

$$\mathbb{P}[N > n] \leqslant ne^{1-f(x^{*})} + o\left(\mathbb{P}[N > n]\right) \leqslant ne^{1-x_1} + o\left(\mathbb{P}[N > n]\right) < 2ne^{1-x_1},$$

and therefore, for n large enough,

$$\log\left(\mathbb{P}[N > n]^{-1}\right) \geqslant \frac{n}{e^{(\log n - (1-\epsilon)(\log n)^{1/\delta})^{1/\delta}}} - \log(2n) - 1,$$

which implies

$$\log\left(\log\left(\mathbb{P}[N > n]^{-1}\right)\right) - \log n + (\log n)^{\frac{1}{\delta}} \gtrsim \frac{1}{\delta}\left(\log n\right)^{\frac{2}{\delta}-1}. \tag{2.34}$$

Next, we prove the *lower bound*. By using the same arguments as in the proof of the lower bound for Theorem 2.3, we obtain, for n large enough,

$$\log\left(\mathbb{P}[N > n]^{-1}\right) \leqslant x_0 + \frac{1}{1-\epsilon}\log\left(\Phi\left(\frac{(1+\epsilon)n}{x_0}\right)\right) = x_0 + \frac{1}{1-\epsilon}e^{\left(\log\left(\frac{(1+\epsilon)n}{x_0}\right)\right)^{\delta}},$$

which, by choosing $x_0 = (1+\epsilon)ne^{-\left(\log n - (\log n)^{1/\delta}\right)^{1/\delta}}$, passing $n \to \infty$ and then $\delta \to 0$, yields

$$\log\left(\log\left(\mathbb{P}[N > n]^{-1}\right)\right) - \log n + (\log n)^{\frac{1}{\delta}} \lesssim \frac{1}{\delta}\left(\log n\right)^{\frac{2}{\delta}-1}. \tag{2.35}$$

Finally, combining (2.34) and (2.35) finishes the proof.

Theorem 2.4 *If $\log(\bar{F}^{-1}(x)) \sim e^{R_{\gamma}\left(\bar{G}^{-1}(x)\right)}$, where $R_{\gamma}(\cdot)$ is regularly varying with index $\gamma > 0$, then,*

$$\log \mathbb{P}[N > n]^{-1} \sim \frac{n}{R_{\gamma}^{\leftarrow}(\log n)}, \tag{2.36}$$

*where $R_{\gamma}^{\leftarrow}(\cdot)$ is the asymptotic inverse of $R_{\gamma}(\cdot)$ as defined in Theorem 1.5.12 on p. 28 of [**?**].*

Remark 12 Note that the functional form in (2.36) is different from the one in (2.25) that describes the Weibull case. In principle, one could study the situations when $\Phi(\cdot)$ grows faster than three exponential scales, which would make the distributions of N even closer to the exponential one. However, from a practical point of view, these cases will basically be indistinguishable from the exponential distribution and, thus, we omit these derivations.

Proof: First, we prove the *upper bound*. Following the same approach as in the proof of Theorem 2.2, we obtain, for $0 < \epsilon < 1$ and $y > 0$,

$$\mathbb{P}[N > n] \leqslant \sum_{k=0}^{\lfloor n/y \rfloor - 1} e^{-k - \frac{1}{1+\epsilon} e^{R_\gamma\left(\frac{n}{k+1}\right)}} + o\left(\mathbb{P}[N > n]\right). \tag{2.37}$$

By using the same argument as in (2.13), we can choose $R_\gamma^*(x) = \gamma \int_1^x R(s)s^{-1}ds, x \geqslant 1$ and $R_\gamma^*(\cdot)$ is absolutely continuous, strictly increasing with an inverse $R_\gamma^{\leftarrow}(\cdot)$. Theorem 1.5.12 on p. 28 and Proposition 1.5.14 on p. 29 of [?] implies that $R_\gamma^{\leftarrow}(\cdot)$ is regularly varying with index $1/\gamma$ and is also the asymptotic inverse of $R_\gamma(\cdot)$. Therefore, there exists $y > 0$ such that for $0 < x < n/y$,

$$x + \frac{1}{1+\epsilon} e^{R_\gamma\left(\frac{n}{x}\right)} \geqslant x + \frac{1}{1+\epsilon} e^{(1-\epsilon)R_\gamma^*\left(\frac{n}{x}\right)}.$$

Suppose that $f(x) \triangleq x + e^{(1-\epsilon)R_\gamma^*\left(\frac{n}{x}\right)}/(1+\epsilon)$ reaches the minimum at x^*, and note that

$$f'(x) = 1 - \frac{1-\epsilon}{1+\epsilon} e^{(1-\epsilon)R_\gamma^*\left(\frac{n}{x}\right)} \left(R_\gamma^*\left(\frac{n}{x}\right)\right)' \frac{n}{x^2} = 1 - \frac{1-\epsilon}{1+\epsilon} e^{(1-\epsilon)R_\gamma^*\left(\frac{n}{x}\right)} \frac{\gamma R_\gamma^*\left(\frac{n}{x}\right)}{x}$$

is an increasing function for x on $(0, n/y)$. Now, defining

$$x_1 \triangleq \frac{n}{R_\gamma^{\leftarrow}\left(\frac{1}{1-\epsilon}\left(\log n - (1-\epsilon)\left(1+\frac{1}{\gamma}\right)\log\log n\right)\right)}, \tag{2.38}$$

it is easy to check that, for all n large enough, $f'(x_1)$ is equal to

$$1 - \frac{\gamma R_\gamma^{\leftarrow}\left(\frac{1}{1-\epsilon}\left(\log n - (1-\epsilon)\left(1+\frac{1}{\gamma}\right)\log\log n\right)\right)\left(\log n - (1-\epsilon)\left(1+\frac{1}{\gamma}\right)\log\log n\right)}{(1+\epsilon)(\log n)^{(1-\epsilon)(1+\frac{1}{\gamma})}} < 0,$$

which implies that $f(x)' < 0$ for $0 < x < x_1$ and n large. Thus, the minimum point x^* satisfies

$$x^* > x_1. \tag{2.39}$$

Combining (2.37) and (2.39) yields, for n large enough,

$$\mathbb{P}[N > n] \leqslant \frac{n}{y} e^{1-f(x^*)} + o\left(\mathbb{P}[N > n]\right) \leqslant \frac{2n}{y} e^{1-x_1},$$

resulting in

$$\log\left(\mathbb{P}[N > n]^{-1}\right) \geqslant \frac{n}{R_\gamma^{\leftarrow}\left(\frac{1}{1-\epsilon}\left(\log n - (1-\epsilon)\left(1+\frac{1}{\gamma}\right)\log\log n\right)\right)} - \log\left(\frac{2n}{y}\right) - 1.$$

Therefore, passing $n \to \infty$ and then $\epsilon \to 0$ in the preceding inequality yields

$$\log \mathbb{P}[N > n]^{-1} \gtrsim \frac{n}{R_\gamma^{\leftarrow}(\log n)}. \tag{2.40}$$

Next, we prove the *lower bound*. By using the same arguments as in the proof of the lower bound for Theorem 2.3, we obtain, for n large enough,

$$\log\left(\mathbb{P}[N>n]^{-1}\right) \leqslant x_0 + \frac{1}{1-\epsilon}\log\left(\Phi\left(\frac{(1+\epsilon)n}{x_0}\right)\right) = x_0 + \frac{1}{1-\epsilon}e^{R_\gamma\left(\frac{(1+\epsilon)n}{x_0}\right)},$$

which, by choosing

$$x_0 = \frac{(1+\epsilon)n}{R_\gamma^{\leftarrow}\left((1-\epsilon)\log n - \frac{1}{\gamma}\log\log n\right)},$$

and noting that $R_\gamma\left(R_\gamma^{\leftarrow}(x)\right) \leqslant x/(1-\epsilon)$ for all x large enough, yields, for n large,

$$\log\left(\mathbb{P}[N>n]^{-1}\right) \leqslant x_0 + \frac{n}{(1-\epsilon)(\log n)^{\frac{1}{(1-\epsilon)\gamma}}}.$$

The preceding inequality implies

$$\log\left(\mathbb{P}[N>n]^{-1}\right) \lesssim \frac{n}{R_\gamma^{\leftarrow}(\log n)}. \tag{2.41}$$

Finally, combining (2.40) and (2.41) finishes the proof. □

2.2 Asymptotics of the Total Transmission Time T

In this subsection, we compute the asymptotics of the total transmission time T based on the previous results on $\mathbb{P}[N>n]$. Our proving technique involves the relationship between N and T described in (1.1) and the classical large deviation results. Theorem 2.5 and Theorem 2.6 characterize the exact asymptotics and logarithmic asymptotics for the very heavy case, respectively, and Theorem 2.7 derives the result for the moderate heavy (Weibull) case. Interestingly, we want to point out that, unlike Theorems 2.5 and 2.6 requiring no conditions on A (Theorem 2.5 needs $\mathbb{E}[A]<\infty$), the minimum conditions needed for Theorem 2.7, as shown by Proposition 2.7, basically involve a balance between the tail decays of $\mathbb{P}[A>x]$ and $\mathbb{P}[L>x]$.

Similarly, the corresponding results on T can be derived for the other statements on N, e.g., Propositions 2.2, 2.3, 2.4, 2.5, and Theorem 2.4. But, to avoid lengthy expositions and repetitions, we omit this derivations. In the following, let $\vee \equiv \max$.

Theorem 2.5 *If $\mathbb{E}\left[U^{(\alpha\vee 1)+\theta}\right]<\infty$, $\mathbb{E}\left[A^{1+\theta}\right]<\infty$ and $\mathbb{E}\left[L^{\alpha+\theta}\right]<\infty$ for some $\theta>0$, then, under the same conditions as in Theorem 2.1 i), i.e., $\bar{F}^{-1}(x)\sim\Phi\left(\bar{G}^{-1}(x)\right)$ with $\Phi(x)$ being regularly varying of index $\alpha>0$, we obtain, as $t\to\infty$,*

$$\mathbb{P}[T>t] \sim \frac{\Gamma(\alpha+1)(\mathbb{E}[U+A])^\alpha}{\Phi(t)}. \tag{2.42}$$

Remark 13 Note that $\mathbb{E}\left[L^{\alpha+\theta}\right]<\infty$ is basically a minimum condition for $\alpha>1$ since $\mathbb{E}\left[L^{\alpha-\theta}\right]=\infty$ implies $\mathbb{E}\left[T^{\alpha-\theta}\right]=\infty$ because of $T\geqslant L$, which would contradict (2.42).

The **proof** is presented in Subsection 4.6.

Theorem 2.6 *Under the same conditions of Theorem 2.2, i.e., the eventually non-decreasing function $\Phi(x)\triangleq e^{l(x)}$ satisfies (2.1) where $l(x)$ is slowly varying with*

$$\lim_{x\to\infty}\frac{l\left(\frac{x}{l(x)}\right)}{l(x)} = 1, \tag{2.43}$$

and in addition, if $\mathbb{P}[L > x] = O\left(\Phi(x)^{-(\delta+1)}\right)$ *and* $\mathbb{P}[U > x] = O\left(\Phi(x)^{-(\delta+1)}\right)$, $\delta > 0$, *then, we obtain*

$$\lim_{t\to\infty} \frac{\log\left(\mathbb{P}[T > t]^{-1}\right)}{\log(\Phi(t))} = 1. \tag{2.44}$$

Remark 14 This result implies parts (1:1), (2:1) and (2:2) of Theorem 2.1 in [**?**] and extends Theorem 2 in [**?**]. Furthermore, it shows that, if $\log \mathbb{P}[L > x]^{-1} \approx \alpha \log \mathbb{P}[A > x]^{-1}$, meaning that the hazard functions of L and A are asymptotically linear, the distribution tails of the number of transmissions and total transmission time are essentially power laws. Thus, the system can exhibit high variations and possible instability, e.g., when $0 < \alpha < 2$, the transmission time has an infinite variance and, when $0 < \alpha < 1$, it does not even have a finite mean.

Remark 15 It is easy to understand that, if the data sizes (e.g., files, packets) follow heavy-tailed distributions, the total transmission time is also heavy-tailed. However, from these two theorems, we see that even if the distributions of the data and channel characteristics are highly concentrated, e.g., when they are asymptotically proportional on the logarithmic scale (see Corollary 2.2 in Subsection 2.1.1), the heavy-tailed transmission delays can still arise.

The **proof** is presented in Subsection 4.7.

Theorem 2.7 *Under the same conditions of Theorem 2.3, i.e., the eventually non-decreasing function* $\Phi(x) \triangleq e^{R_\beta(x)}$ *satisfies (2.1) where* $R_\beta(x) = x^\beta l(x)$, $\beta > 0$ *is regularly varying with* $l(x)$ *satisfying*

$$\lim_{x\to\infty} \frac{l\left(\left(\frac{x}{l(x)}\right)^{\frac{1}{1+\beta}}\right)}{l(x)} = 1, \tag{2.45}$$

and in addition, if $\mathbb{E}[A] < \infty$, $\mathbb{P}[U > x] = O\left(e^{-(\log\Phi(x))^{(1+\delta)/(\beta+1)}}\right)$, $\delta > 0$, *and* $\mathbb{P}[L > x] = O\left(e^{-x^\xi}\right)$, $\mathbb{P}[A > x] = O\left(e^{-x^\zeta}\right)$ *with* $\xi > \beta/(\beta+1), \zeta \geqslant 0$ *satisfying* $(1-\zeta)\beta < \xi$, *then, we obtain*

$$\lim_{n\to\infty} \frac{\log\left(\mathbb{P}[T > t]^{-1}\right)}{(\log\Phi(t))^{\frac{1}{\beta+1}}} = \frac{\beta^{\frac{1}{\beta+1}} + \beta^{-\frac{\beta}{\beta+1}}}{(\mathbb{E}[A+U])^{\frac{\beta}{\beta+1}}}. \tag{2.46}$$

Remark 16 This theorem implies part (1:2) of Theorem 2.1 in [**?**], and provides a more precise logarithmic asymptotics instead of a double logarithmic limit. Furthermore, it is easy to check that the condition $(1-\zeta)\beta < \xi$ holds in two special cases: (i) if $\zeta \geqslant \beta/(\beta+1)$ and $\xi > \beta/(\beta+1)$ or (ii) if $\xi > \beta$ and $\zeta = 0$ (assuming no conditions for $\mathbb{P}[A > x]$ beyond $\mathbb{E}[A] < \infty$).

The **proof** is presented in Subsection 4.8. Basically, the condition $(1-\zeta)\beta < \xi$ (or equivalently $\xi/(\xi+1-\zeta) > \beta/(\beta+1)$) is needed since the following proposition shows that $\mathbb{P}[T > t]$ could have a heavier tail than predicted by (2.46) if $(1-\zeta)\beta > \xi$.

Proposition 2.7 *If* $\mathbb{P}[L > x] = e^{-x^\xi}$ *and* $\mathbb{P}[A > x] = e^{-x^\zeta}$ *with* $0 < \xi, \zeta < 1$, *then, as* $t \to \infty$,

$$\mathbb{P}[T > t] \gtrsim e^{-2t^{\xi/(\xi+1-\zeta)}}.$$

Proof: It is easy to see that, for $\delta, y > 0$,

$$\begin{aligned}\mathbb{P}[T > t] &\geqslant \mathbb{P}\left[T > t, y < A_i < (1+\delta)y, 1 \leqslant i \leqslant \frac{t}{y}, L > (1+\delta)y\right] \\ &\geqslant \left(\mathbb{P}[y < A < (1+\delta)y]\right)^{\frac{t}{y}} \mathbb{P}[L > (1+\delta)y],\end{aligned}$$

which, by noting that $\mathbb{P}[A > x] = e^{-x^\zeta}$ with $\zeta > 0$, yields

$$\mathbb{P}\left[T > t\right] \gtrsim \left(\mathbb{P}[A > y]\right)^{\frac{t}{y}} \mathbb{P}[L > (1+\delta)y] = e^{-\left(\frac{t}{y}y^\zeta + y^\xi\right)}. \tag{2.47}$$

Choosing $y = t^{1/(\xi+1-\zeta)}$ finishes the proof. □

3 Engineering Implications

As already stated in the introduction, retransmissions are the integral component of many modern networking protocols on all communication layers from the physical to the application one. In our recent work [?, ?, ?, ?], we have shown that these protocols may result in heavy-tailed (e.g., power law) delays even if all the system components are light-tailed (superexponential). More specifically, from an engineering perspective, our main discovery is the matching between the statistical characteristics of the channel and transmitted data (e.g., packets). Basically, one can expect good or bad delay performance measured by the existence of α-moments for N and T if $\alpha \log \mathbb{P}[A > x] > \log \mathbb{P}[L > x]$ or $\alpha \log \mathbb{P}[A > x] < \log \mathbb{P}[L > x]$, respectively. Note that, if $\alpha < 1$, then the system could experience zero throughput.

On the network application layer, most of us have experienced the connection failures while downloading a large file from the Internet. This issue has been already recognized in practice where software for downloading sizable documents was developed that would save the intermediate data (checkpoints) and resume the download from the point when the connection was broken. However, our results emphasize that, in the presence of frequently failing connections, the long delays may arise even when downloading relatively small documents. Hence, we argue that one may need to adopt the application layer software for the wireless environment by introducing checkpoints even for small to moderate size documents.

Furthermore, on the physical layer, it is well known that wireless links, especially for low-powered sensor networks, have higher error rates than the wired counterparts. This may result in large delays on the data link layer due to the (IP) packet variability and channel failures. Therefore, our results suggest that packet fragmentation techniques need to be applied with special care since: if the packets are too small, they will mostly contain the packet header, which can limit the useful throughput; if the packets are too large, power law delays can deteriorate the quality of transmission. When the codewords, the basic units of packets in the physical layer, are much smaller than the maximum size of the packets, our results show that the number of retransmissions could be power law, which challenges the traditional model that assumes a geometric number of retransmissions. We believe that short codewords are realistic assumption for sensor networks, where complicated coding schemes are unlikely since the nodes have very limited computational power. In reality, packet sizes may have an upper limit (e.g., WaveLAN's maximum transfer unit is 1500 bytes), this situation may result in truncated power law distributions for T and N in the main body with a stretched (exponentiated) support in relation to the support of L (see Example 3 in Section IV of [?]) and, thus, may result in very long, although, exponentially bounded delays. The impact of truncated heavy-tailed distributions on queueing behavior was quantified in [?].

On the medium access control layer, ALOHA is a widely used protocol that provides a contention management scheme for multiple users sharing the same medium. Once a user detects a collision, it will back off for a random (exponential) period of time before trying to retransmit the collided packet. Due to its simplicity and distributed nature, ALOHA is the basis of many other protocols, such as CSMA/CD. We discovered a new phenomenon in [?] that a basic finite population ALOHA model with variable size (exponential) packets is

characterized by power law transmission delays, possibly even resulting in zero throughput; see Theorem 1 and Example 1 in [**?**] that characterizes and illustrates the observation respectively. This power law effect might be diminished, or perhaps eliminated, by reducing the variability of packets. However, we also show in [**?**] that even a slotted (synchronized) ALOHA with packets of constant size can exhibit power law delays when the number of active users is random; see Theorem 2 and Example 2 in [**?**]. The ALOHA system is a generalization of our study in this paper, since, informally, it can be viewed as the state dependent version of the model considered here where the distributions of L and A depend on the state of the system.

On the transport layer, most of the network protocols (e.g., TCP) use end-to-end acknowledgements for packets as an error control strategy. Namely, once the packet sent from the sender to the receiver is lost due to, e.g., finite buffers or link failures, this packet will be retransmitted by the sender. Furthermore, the number of hops that a packet traverses on its path to the destination is random, e.g., an end-user that is surfing the Web might download documents from diverse web sites. Our recent work in [**?**] shows that this acknowledgement mechanism, jointly with the random number of hops, may result in heavy-tailed (e.g., power law) delays. To illustrate this phenomenon, we consider the following basic example. Assume that a single data unit (packet) needs to traverse a random geometric number L of hops before reaching the destination, $\mathbb{P}[L > n] = e^{-pn}, p > 0$. Next, assume that in each hop the packet can independently (independent of L as well) be lost with probability $1 - e^{-q}, q > 0$. When a packet is lost, it is retransmitted by the sender and this procedure continues until the packet reaches its destination. Then, it is easy to see that, in conjunction with Remark 3 after Theorem 2.1, the number N of retransmissions that the sender needs to perform satisfies

$$\frac{e^{-p}\Gamma(1+p/q)}{n^{p/q}} \leqslant \mathbb{P}[N > n] \leqslant \frac{e^{p}\Gamma(1+p/q)}{n^{p/q}}.$$

Similarly, assuming that in each hop a packet is processed for one unit of time, we can derive that the distribution of the total transmission time T satisfies $\log(\mathbb{P}[T > t]) \sim -(p/q)\log t$.

Furthermore, when the cause of losses is due to the finiteness of buffers, i.e., a packet is lost when it sees a full buffer upon its arrival, the preceding general setup can be more precisely modeled as a sequence of random number L of tandem queues [**?**]. More specifically, we consider L tandem $\cdot/M/1/b$ queues with each queue being able to accommodate up to (finitely many) b packets; M stands for exponential (memoryless) service times. This model can be shown to result in heavy-tailed delays under quite general assumptions on cross traffic, network topology and routing scenarios. However, for simplicity we only present the following example. As depicted in Figure 2, suppose that the single packet, as well as the cross traffic, is sent sequentially through a chain of finite buffer queues with capacity b. Also, we assume that the cross traffic flows are i.i.d Poisson processes and the service requirements needed for processing different packets and the same packet at different queues are i.i.d. exponential random variables. Furthermore, the sender tries to transmit a single packet through the sequence of queues, and if the packet is lost, the packet will be immediately retransmitted by the sender. Then, regardless of how many hops the cross traffic flows traverse before leaving the system, the distribution of the number of retransmissions N satisfies the following Theorem 3.1.

Theorem 3.1 *If the limit $p \triangleq \lim_{n\to\infty} \log(\mathbb{P}[L > n])/n < 0$ exists, then, there exists $0 < \alpha_1 \leqslant \alpha_2 < \infty$, such that*

$$-\alpha_2 \leqslant \varliminf_{n\to\infty} \frac{\log \mathbb{P}[N > n]}{\log n} \leqslant \varlimsup_{n\to\infty} \frac{\log \mathbb{P}[N > n]}{\log n} \leqslant -\alpha_1.$$

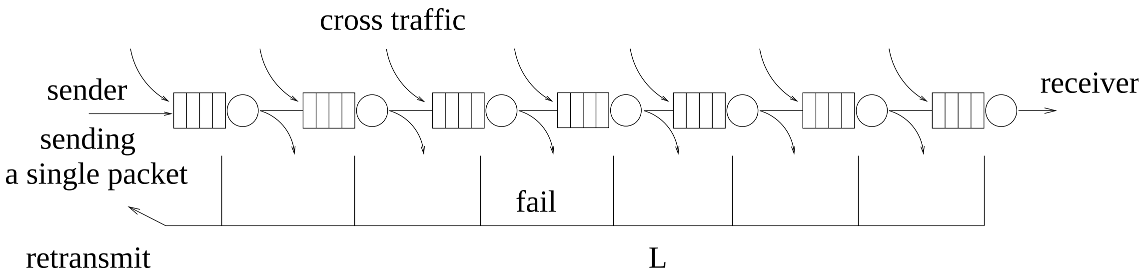

Figure 2: Tandem $\cdot/M/1/b$ queues with finite buffers

Remark 17 Note that the same result can be easily derived for T. Furthermore, due to the generality of our argument, the proof of this theorem can be applied to much more complicated routing schemes, network topologies and cross traffic conditions, e.g., with routing loops. However, such generalizations, except for complex notation, do not bring new insights, and therefore, we only study the current simple example. Further study of this model will be available in [**?**].

Proof: Number the sequence of queues from 1 to L sequentially. Recall that the packet is lost when it sees a full buffer upon its arrival. In order to prove the theorem, we construct two systems with independent loss probabilities at different queues that provide upper and lower bounds on the loss probabilities for the considered packet.

First, we prove the *lower bound*. Construct a system that empties queue $i+1$ whenever the considered packet begins receiving service in queue i ($1 \leqslant i < L$). Denote by C_i the event that this packet is lost when arriving at queue $i+1$. From the procedure of this construction and the memoryless property, it is easy to see that $\{C_i\}_{1 \leqslant i \leqslant L}$ are i.i.d. conditional on L. Thus, we obtain, for $n_0 > 0$,

$$\mathbb{P}[N > n] \geqslant \mathbb{E}\left[\left(1 - \prod_{i=1}^{L}(1 - \mathbb{P}[C_i])\right)^n \middle| L\right] \geqslant \mathbb{E}\left[\left(1 - (1 - \mathbb{P}[C_1])^L\right)^n \mathbf{1}(L > n_0)\right]. \tag{3.1}$$

Then, we construct a continuous random variable L^* with $\mathbb{P}[L^* > x] = e^{-2px}, x \geqslant 0$, and choose n_0 large enough such that $\mathbb{P}[L^* > x] \leqslant \mathbb{P}[L > x]$ for all $x > n_0$. Therefore, by using stochastic dominance and replacing L with L^* in (3.1), we obtain

$$\mathbb{P}[N > n] \geqslant \mathbb{E}\left[\left(1 - (1 - \mathbb{P}[C_1])^{L^*}\right)^n\right] - \left(1 - (1 - \mathbb{P}[C_1])^{n_0}\right)^n, \tag{3.2}$$

which, by setting $\bar{G}(x) = (1 - \mathbb{P}[C_1])^x$, $\bar{F}(x) = \mathbb{P}[L^* > x]$ and applying Theorem 2.2, yields, for some $\alpha_2 > 0$,

$$\varliminf_{n \to \infty} \frac{\log \mathbb{P}[N > n]}{\log n} \geqslant -\alpha_2. \tag{3.3}$$

Note that this line of argument can be used to rigorously prove Remark 6.

Next, we prove the *upper bound*. Construct a system that empties queue $i+1$ whenever the considered packet begins receiving service in queue i ($1 \leqslant i < L$). Then, using this construction and similar arguments as in the proof of the lower bound, we can easily prove that there exists $\alpha_1 > 0$ such that

$$\varlimsup_{n \to \infty} \frac{\log \mathbb{P}[N > n]}{\log n} \leqslant -\alpha_1,$$

which, in conjunction with (3.3), finishes the proof. □

Finally, we would like to point out that, in addition to the preceding applications in communication networks [**?**, **?**, **?**, **?**] and job processing on machines with failures [**?**, **?**], the model studied in this paper may represent a basis for understanding more complex failure prone systems, e.g., see the recent study on parallel computing in [**?**].

In conclusion, we would like to emphasize that, in practice, our results provide an easily computable benchmark for measuring the tradeoff between the data statistics and channel characteristics that permits/prevents satisfactory transmission.

4 Proofs

4.1 Proof of Proposition 1.1

As stated earlier in Subsection 1.1, the proof of this proposition was originally presented in Lemma 1 of [**?**] and, we repeat it here for reasons of completeness.

Proof: Note that for any $\delta > 0$, there exists $t_\delta > 0$ such that, for all $0 < t < t_\delta$,

$$1 - t \geqslant e^{-\delta} e^{-t}.$$

Therefore, we can choose x_δ large enough, such that $1 - \bar{G}(x) \geqslant e^{-\delta} e^{-\bar{G}(x)}$ for all $x > x_\delta$. Then,

$$\begin{aligned} e^{\epsilon n}\mathbb{P}[N > n] &\geqslant e^{\epsilon n}\mathbb{E}\left[\left(1 - \bar{G}(L)\right)^n \mathbf{1}(L \geqslant x_\delta)\right] \geqslant e^{\epsilon n}\mathbb{E}\left[e^{n\delta} e^{-n\bar{G}(L)}\mathbf{1}(L \geqslant x_\delta)\right] \\ &\geqslant \left(e^{\epsilon - \bar{G}(x_\delta) - \delta}\right)^n \bar{F}(x_\delta). \end{aligned}$$

Thus, by selecting δ small enough and x_δ large enough, we can always make $e^{\epsilon - \bar{G}(x_\delta) - \delta} > 1$, and, by passing $n \to \infty$, we complete the proof of (1.2).

Next, we prove the corresponding result for T. Suppose that $\bar{G}(x_0) > 0$ for some $x_0 > 0$; otherwise, T will be infinite, which yields (1.3) immediately. We can always find $x_1 > x_0 > 0$, such that i.i.d. random variables $X_i \triangleq x_0 \mathbf{1}(x_0 < A_i < x_1)$ satisfy $0 < \mathbb{E}X_1 < \infty$. Now, for any $\zeta > 0$,

$$\begin{aligned} \mathbb{P}[T > t] &= \mathbb{P}\left[\sum_{i=1}^{N-1}(U_i + A_i) + L > t\right] \geqslant \mathbb{P}\left[\sum_{i=1}^{N-1} A_i \mathbf{1}(x_0 < A_i < x_1) > t\right] \\ &\geqslant \mathbb{P}\left[\sum_{i=1}^{N-1} X_i > t\right] \geqslant \mathbb{P}\left[\sum_{i=1}^{N-1} X_i > t, N \geqslant \frac{t(1+\zeta)}{\mathbb{E}X_1}\right] \\ &\geqslant \mathbb{P}\left[N > \frac{t(1+\zeta)}{\mathbb{E}X_1} + 1\right] - \mathbb{P}\left[\sum_{i=1}^{N-1} X_i \leqslant t, N > \frac{t(1+\zeta)}{\mathbb{E}X_1} + 1\right] \\ &\triangleq I_1 - I_2. \end{aligned} \tag{4.1}$$

Since, for $\bar{X}_i \triangleq \mathbb{E}[X_i] - X_i$,

$$I_2 \leqslant \mathbb{P}\left[\sum_{i \leqslant t(1+\zeta)/\mathbb{E}X_1} X_i \leqslant t\right] = \mathbb{P}\left[\sum_{i \leqslant t(1+\zeta)/\mathbb{E}X_1} \bar{X}_i \geqslant \zeta t\right], \tag{4.2}$$

it is well known (e.g., see Example 1.15 of [**?**]) that there exists $\eta > 0$, such that

$$I_2 \leqslant e^{-\eta t}. \tag{4.3}$$

Therefore, by (1.2), (4.1) and (4.3), we obtain that for all $0 < \epsilon < \eta$,

$$e^{\epsilon t}\mathbb{P}[T > t] \to \infty \ \text{ as } t \to \infty,$$

implying that (1.3) holds for any $\epsilon > 0$. □

4.2 Proof of Proposition 2.1

Proof: If $\log(\Phi(x))$ is slowly varying, then, for any $0 < \delta < \epsilon$, there exists $x_\delta > 0$ such that $\log(\Phi(x)) < x^\delta$ for all $x > x_\delta$. By using the condition (2.1), or equivalently (2.21), we obtain, for n large enough,

$$\begin{aligned}
\mathbb{P}[N > n] &= \mathbb{E}[(1 - \bar{G}(L))^n] \\
&\geqslant \left(1 - \frac{1}{n}\right)^n \mathbb{P}\left[\bar{G}(L) \leqslant \frac{1}{n}\right] \\
&\geqslant \left(1 - \frac{1}{n}\right)^n \mathbb{P}\left[\Phi^{\leftarrow}\left(\bar{F}^{-(1+\epsilon)}(L)\right) \geqslant n\right] \\
&\geqslant \left(1 - \frac{1}{n}\right)^n e^{-x^\delta},
\end{aligned}$$

where we use the fact that for x_ϵ chosen in (2.21) one can always select n large enough such that $\{\bar{G}(L) \leqslant 1/n\} \subset \{L > x_\epsilon\}$. Therefore, we obtain,

$$0 \leqslant \varlimsup_{n\to\infty} \frac{-\log \mathbb{P}[N > n]}{n^\epsilon} \leqslant \lim_{n\to\infty} \frac{-1 + n^\delta}{n^\epsilon} = 0,$$

which proves the proposition. □

4.3 Continuation of the proof of Theorem 2.1

Proof: Now, we prove the *lower bound*. For $K > 0$ and x_ϵ selected in (2.9), choosing $x_n > x_\epsilon$ with $\Phi^{\leftarrow}\left((1 - \epsilon)\bar{F}(x_n)\right) = n/K$, we obtain, for large n,

$$\begin{aligned}
\mathbb{P}[N > n] &= \mathbb{E}\left[\left(1 - \bar{G}(L)\right)^n\right] \\
&\geqslant \mathbb{E}\left[\left(1 - \bar{G}(L)\right)^n \mathbf{1}(L > x_n)\right] \\
&\geqslant \mathbb{E}\left[\left(1 - \frac{1}{\Phi^{\leftarrow}\left((1 - \epsilon)V^{-1}\right)}\right)^n \mathbf{1}(V < \bar{F}(x_n))\right],
\end{aligned}$$

which, by letting $z = n/\Phi^{\leftarrow}\left((1 - \epsilon)V^{-1}\right)$, yields

$$\mathbb{P}[N > n]\Phi(n) \geqslant \int_0^K \left(1 - \frac{z}{n}\right)^n \frac{\Phi(n)}{\Phi(n/z)} \frac{\Phi'(n/z)}{\Phi(n/z)} \frac{(1 - \epsilon)n}{z^2} dz. \tag{4.4}$$

From (4.4), by using the same approach as in deriving (2.17), we obtain, as $n \to \infty$,

$$\mathbb{P}[N > n]\Phi(n) \sim \int_0^K (1 - \epsilon)\alpha e^{-z} z^{\alpha - 1} dz,$$

which, by passing $K \to \infty$ and $\epsilon \to 0$, yields

$$\mathbb{P}[N > n]\Phi(n) \gtrsim \int_0^\infty \alpha e^{-z} z^{\alpha-1} dz = \Gamma(\alpha + 1). \tag{4.5}$$

Combining (2.18) and (4.5) completes the proof of (2.4).

Then, we proceed with proving (2.5). First, we prove the *lower bound*. Since $\Phi(x)$ is eventually non-decreasing, we obtain the inequality presented in (2.9) again, and therefore, for n large enough and $\epsilon > 0$,

$$\begin{aligned}
\mathbb{P}[N > n] &= \mathbb{E}[(1 - \bar{G}(L))^n] \\
&\geqslant \left(1 - \frac{\epsilon}{n}\right)^n \mathbb{P}\left[\bar{G}(L) \leqslant \frac{\epsilon}{n}\right] \\
&\geqslant \left(1 - \frac{\epsilon}{n}\right)^n \mathbb{P}\left[\Phi^{\leftarrow}\left((1-\epsilon)\bar{F}^{-1}(L)\right) \geqslant \frac{n}{\epsilon}\right] \\
&\geqslant \left(1 - \frac{\epsilon}{n}\right)^n \frac{1-\epsilon}{\Phi\left(\frac{n}{e}\right)},
\end{aligned}$$

implying

$$\varliminf_{n\to\infty} \mathbb{P}[N > n]\Phi(n) \geqslant \varliminf_{n\to\infty} \left(1 - \frac{\epsilon}{n}\right)^n \frac{(1-\epsilon)\Phi(n)}{\Phi\left(\frac{n}{\epsilon}\right)},$$

which, by passing $\epsilon \to 0$, yields

$$\varliminf_{n\to\infty} \mathbb{P}[N > n]\Phi(n) \geqslant 1. \tag{4.6}$$

Next, we prove the *upper bound*. Using a similar approach that derived (2.10), we obtain

$$\begin{aligned}
\mathbb{P}[N > n] &\leqslant \mathbb{E}\left[e^{-\frac{n}{\Phi^{\leftarrow}((1+\epsilon)V^{-1})}}\right] + \left(1 - \bar{G}(x_\epsilon)\right)^n \\
&\leqslant \mathbb{P}\left[0 \leqslant \frac{n}{\Phi^{\leftarrow}\left((1+\epsilon)V^{-1}\right)} \leqslant e^m\right] \\
&\quad + \sum_{k=m}^{\lceil \log(\epsilon n) \rceil} e^{-e^k} \mathbb{P}\left[e^k \leqslant \frac{n}{\Phi^{\leftarrow}\left((1+\epsilon)V^{-1}\right)} \leqslant e^{k+1}\right] + o\left(\frac{1}{\Phi(n)}\right) \\
&\leqslant \frac{1+\epsilon}{\Phi\left(\frac{n}{e^m}\right)} + \sum_{k=m}^{\lceil \log(\epsilon n) \rceil} e^{-e^k} \frac{1+\epsilon}{\Phi\left(\frac{n}{e^{k+1}}\right)} + o\left(\frac{1}{\Phi(n)}\right),
\end{aligned}$$

resulting in

$$\mathbb{P}[N > n]\Phi(n) \leqslant \frac{(1+\epsilon)\Phi(n)}{\Phi\left(\frac{n}{e^m}\right)} + \sum_{k=m}^{\lceil \log(\epsilon n) \rceil} e^{-e^k} \frac{(1+\epsilon)\Phi(n)}{\Phi\left(\frac{n}{e^{k+1}}\right)} + o\left(1\right). \tag{4.7}$$

Note that the second term in the right hand side of (4.7) is always finite because of (2.11) and, by passing $n \to \infty$ and then $m \to \infty$ in (4.7), we obtain

$$\varlimsup_{n\to\infty} \mathbb{P}[N > n]\Phi(n) \leqslant 1. \tag{4.8}$$

Combining (4.6) and (4.8) finishes the proof of (2.5). $\square$

4.4 Proof of Proposition 2.4

Proof: First, we prove the *lower bound*. By recalling the condition (2.3), or equivalently (2.9), and using $1-x \geqslant e^{-(1+\epsilon)x}$ for x small enough, we obtain, for n large enough and $x_0 > 0$,

$$\begin{aligned}\mathbb{P}[N>n] &\geqslant \mathbb{E}[(1-\bar{G}(L))^n \mathbf{1}(L \geqslant x_\epsilon)] \geqslant \mathbb{E}[e^{-(1+\epsilon)\bar{G}(L)n}\mathbf{1}(L \geqslant x_\epsilon)] \\ &\geqslant \mathbb{E}\left[e^{-\frac{(1+\epsilon)n}{\Phi^{\leftarrow}((1-\epsilon)V^{-1})}}\mathbf{1}(V \leqslant \bar{F}(x_\epsilon))\right] \geqslant e^{-x_0}\mathbb{P}\left[\frac{(1+\epsilon)n}{\Phi^{\leftarrow}((1-\epsilon)V^{-1})} \leqslant x_0, V \leqslant \bar{F}(x_\epsilon)\right] \\ &= e^{-x_0}(1-\epsilon)\Phi\left(\frac{(1+\epsilon)n}{x_0}\right)^{-1} = (1-\epsilon)e^{-x_0-\lambda\left(\log n - \log\left(\frac{x_0}{1+\epsilon}\right)\right)^\delta}, \end{aligned} \tag{4.9}$$

Using the preceding inequality and setting $x_0 = \lambda\delta(\log n)^{\delta-1}$ yields, for n large enough,

$$\begin{aligned}\log \mathbb{P}[N>n]^{-1} - \lambda(\log n)^\delta &\leqslant \lambda\left(\log n - \log\left(\frac{x_0}{1+\epsilon}\right)\right)^\delta - \lambda(\log n)^\delta + x_0 - \log(1-\epsilon) \\ &\leqslant -(1-\epsilon)\lambda\delta\,(\log n)^{\delta-1}\log\left(\lambda\delta(\log n)^{\delta-1}\right) + \lambda\delta(\log n)^{\delta-1},\end{aligned}$$

which, by passing $n\to\infty$ and then $\epsilon \to 0$, results in

$$\log \mathbb{P}[N>n]^{-1} - \lambda(\log n)^\delta \lesssim -\lambda\delta(\delta-1)(\log\log n)(\log n)^{\delta-1}. \tag{4.10}$$

Next, we prove the *upper bound*. Following the same approach as in the proof of Theorem 2.2, we obtain, for large n and $y = \lambda(\log n)^\delta - \lambda\delta(\delta-1)\log\log n(\log n)^{\delta-1}$,

$$\begin{aligned}\mathbb{P}[N>n] &\leqslant \sum_{k=0}^{y-1} e^{-k}\mathbb{P}\left[k \leqslant \frac{n}{\Phi^{\leftarrow}((1+\epsilon)V^{-1})} \leqslant k+1\right] + e^{-y} + o\left(\mathbb{P}[N>n]\right) \\ &\leqslant (1+\epsilon)\sum_{k=0}^{y-1} e^{-k-\lambda(\log n - \log(k+1))^\delta} + e^{-y} + o\left(\mathbb{P}[N>n]\right).\end{aligned} \tag{4.11}$$

Suppose that $f(x) = x + \lambda\left(\log n - \log x\right)^\delta$ reaches the minimum at x^* when $1 \leqslant x \leqslant y$. It is easy to check that $f'(x) = 1 - \lambda\delta(\log n - \log x)^{\delta-1}/x$ is monotonically increasing for x in $(0,n)$. Then, by defining

$$x_1 \triangleq \lambda\delta(\log n)^{\delta-1} - (1-\epsilon)\lambda\delta(\delta-1)^2(\log\log n)(\log n)^{\delta-2}, \;\; \epsilon > 0,$$

we obtain, after some easy calculations, for large n,

$$f'(x_1) \geqslant 1 - \frac{(\log n)^{\delta-1} - (1-\epsilon/2)(\delta-1)^2(\log\log n)(\log n)^{\delta-2}}{(\log n)^{\delta-1} - (1-\epsilon)(\delta-1)^2(\log\log n)(\log n)^{\delta-2}} > 0,$$

which implies that $f'(x) > 0$ for $x \geqslant x_1$ and, therefore, $x^* < x_1$ for all $n > n_0$. Hence, by (4.11), we obtain

$$\mathbb{P}[N>n] \leqslant (1+\epsilon)ye^{1-\lambda(\log n - \log x_1)^\delta} + e^{-y} + o\left(\mathbb{P}[N>n]\right), \tag{4.12}$$

which, by recalling the definitions of y and x_1, results in

$$\log \mathbb{P}[N>n]^{-1} - \lambda(\log n)^\delta \gtrsim -(1+\epsilon)\lambda\delta(\delta-1)(\log\log n)(\log n)^{\delta-1}. \tag{4.13}$$

Finally, passing $\epsilon \to 0$ in (4.13) and combining it with (4.10), we finish the proof. □

4.5 Proof of Proposition 2.5

Proof: First, we prove the *lower bound*. Using the same arguments as in the proof of the lower bound for Theorem 2.3, we obtain, for $0 < \epsilon < 1, x_0 > 0$ and n large enough,

$$\log\left(\mathbb{P}[N > n]^{-1}\right) \leqslant x_0 + \frac{1}{1-\epsilon}\log\left(\Phi\left(\frac{(1+\epsilon)n}{x_0}\right)\right) = x_0 + \frac{1}{1-\epsilon}e^{\lambda\left(\log\left(\frac{(1+\epsilon)n}{x_0}\right)\right)^{\delta}}.$$

Setting $x_0 = e^{\lambda(\log n)^{\delta}\left(1-\delta\lambda(\log n)^{\delta-1}\right)}, 1/2 < \delta < 1$ in the preceding inequality yields

$$\log\left(\mathbb{P}[N > n]^{-1}\right) \leqslant e^{\lambda(\log n)^{\delta}\left(1-\delta\lambda(\log n)^{\delta-1}\right)} + \frac{1}{1-\epsilon}e^{\lambda(\log n - \log x_0 + \log(1+\epsilon))^{\delta}},$$

which, by noting that $\lambda(\log n - \log x_0 + \log(1+\epsilon))^{\delta} \leqslant \lambda(\log n)^{\delta}\left(1-(1-\epsilon)\delta\lambda(\log n)^{\delta-1}\right)$ for all n large enough, implies, for n large enough,

$$\log(\log \mathbb{P}[N > n]^{-1}) \leqslant \log\left(1 + \frac{1}{1-\epsilon}\right) + \lambda(\log n)^{\delta}\left(1-(1-\epsilon)\delta\lambda(\log n)^{\delta-1}\right).$$

Passing $\epsilon \to 0$ in the preceding inequality results in

$$\log(\log \mathbb{P}[N > n]^{-1}) - \lambda(\log n)^{\delta} \leqslant -\delta\lambda^2(\log n)^{2\delta-1}. \tag{4.14}$$

Next, we prove the *upper bound*. Following the same approach as in the proof of Theorem 2.2, we obtain

$$\mathbb{P}[N > n] \leqslant \sum_{k=0}^{y-1} e^{-k-\frac{1}{1+\epsilon}e^{\lambda(\log n - \log k)^{\delta}}} + e^{-y} + o\left(\mathbb{P}[N > n]\right). \tag{4.15}$$

Choose $y = e^{\lambda(\log n)^{\delta}\left(1-(1+\epsilon)\delta\lambda(\log n)^{\delta-1}\right)}$ and let $f(x) = x + e^{\lambda(\log n - \log x)^{\delta}}/(1+\epsilon)$. Since $f'(x) = 1 - e^{\lambda(\log n - \log x)^{\delta}}/((1+\epsilon)x)$ is an increasing function for x in $(0, n)$, it is easy to see that, for all $0 < x \leqslant y$ and n large enough,

$$f'(x) \leqslant 1 - \frac{e^{\lambda(\log n - \log y)^{\delta}}}{(1+\epsilon)y} \leqslant 1 - \frac{e^{\lambda(\log n)^{\delta}\left(1-\delta\lambda(\log n)^{\delta-1}\right)}}{(1+\epsilon)e^{\lambda(\log n)^{\delta}\left(1-(1+\epsilon)\delta\lambda(\log n)^{\delta-1}\right)}} < 0.$$

Therefore, for $0 \leqslant k \leqslant y$, we obtain

$$e^{-k-\frac{1}{1+\epsilon}e^{\lambda(\log n - \log k)^{\delta}}} \leqslant e^{-y-\frac{1}{1+\epsilon}e^{\lambda(\log n - \log y)^{\delta}}},$$

which, by (4.15), yields

$$\begin{aligned}\mathbb{P}[N > n] &\leqslant ye^{-y-\frac{1}{1+\epsilon}e^{\lambda(\log n - \log y)^{\delta}}} + e^{-y} + o\left(\mathbb{P}[N > n]\right)\\ &\leqslant (y+1)e^{-y} + o\left(\mathbb{P}[N > n]\right),\end{aligned} \tag{4.16}$$

implying

$$\log(\log \mathbb{P}[N > n]^{-1}) - \lambda(\log n)^{\delta} \gtrsim -(1+\epsilon)\delta\lambda^2(\log n)^{2\delta-1}. \tag{4.17}$$

Finally, by passing $\epsilon \to 0$ in (4.17) and combining it with (4.14), we finish the proof. □

4.6 Proof of Theorem 2.5

The proofs are based on large deviation results developed by S. V. Nagaev in [**?**]; specifically, we summarize Corollary 1.6 and Corollary 1.8 of [**?**] in this following lemma.

Lemma 4.1 *Let $X_1, X_2, \cdots X_n$ and X be i.i.d random variables with $\int_{u\geqslant 0} u^s d\mathbb{P}[X < u] < \infty$ and $\mathbb{E}X = 0$.*

If $1 \leqslant s \leqslant 2$, then, there exist finite $y_s, c > 0$ such that for $x > y > y_s$,

$$\mathbb{P}\left[\sum_{i=1}^{n} X_i \geqslant x\right] \leqslant n\mathbb{P}[X > y] + \left(\frac{cn}{xy^{s-1}}\right)^{x/2y}. \tag{4.18}$$

If $s > 2$, then, there exist finite $c > 0$ such that

$$\mathbb{P}\left[\sum_{i=1}^{n} X_i \geqslant x\right] \leqslant \frac{cn}{x^s} + \exp\left(\frac{-x^2}{cn}\right). \tag{4.19}$$

Proof: Please refer to [**?**].

Now, we are ready to prove Theorem 2.5.

Proof: First, we establish the *upper bound*. By recalling Definition 1.1, for any $1/2 > \delta > 0$, we obtain

$$\begin{aligned}
\mathbb{P}[T > t] &= \mathbb{P}\left[\sum_{i=1}^{N-1}(U_i + A_i) + L > t\right] \\
&\leqslant \mathbb{P}\left[\sum_{i=1}^{N}(A_i \wedge L + \mathbb{E}[U]) > (1-2\delta)t\right] + \mathbb{P}\left[\sum_{i=1}^{N}(U_i - \mathbb{E}[U]) > \delta t\right] + \mathbb{P}\left[L > \delta t\right] \\
&\triangleq I_1 + I_2 + I_3.
\end{aligned} \tag{4.20}$$

The condition $\mathbb{E}[L^{\alpha+\epsilon}] < \infty$ implies

$$I_3 \leqslant \frac{\mathbb{E}[L^{\alpha+\theta}]}{(\epsilon t)^{\alpha+\theta}} = O\left(\frac{1}{t^{\alpha+\theta}}\right). \tag{4.21}$$

For I_2, we begin with studying the case of $\alpha > 1$, i.e., when $\mathbb{E}[N] < \infty$. Since N is independent of $\{U_i\}$, by defining $X_i \triangleq U_i - \mathbb{E}[U_i]$, we obtain,

$$I_2 = \sum_{n=1}^{\infty} \mathbb{P}[N = n]\mathbb{P}\left[\sum_{i=1}^{n} X_i > \delta t\right].$$

To evaluate $\mathbb{P}\left[\sum_{i=1}^{n} X_i > \delta t\right]$ in the preceding equality, we need to apply Lemma 4.1, which results in two situations. If $1 < s \triangleq \alpha + \theta \leqslant 2$, using (4.18) with $y = \delta t/2$, we obtain, for all $n \geqslant 1$,

$$\mathbb{P}\left[\sum_{i=1}^{n} X_i > \delta t\right] \leqslant n\mathbb{P}[X_1 > \delta t/2] + \frac{2^{s-1}cn}{\delta^s t^s}, \tag{4.22}$$

implying

$$\begin{aligned}
I_2 &\leqslant \sum_{n=1}^{\infty} \mathbb{P}[N = n]\left(n\mathbb{P}[X_1 > \delta t/2] + \frac{2^{s-1}cn}{\delta^s t^s}\right) \\
&\leqslant \mathbb{E}[N]\mathbb{P}[X_1 > \delta t/2] + \frac{2^{s-1}c\mathbb{E}[N]}{\delta^s t^{\alpha+\theta}} = O\left(\frac{1}{t^{\alpha+\theta}}\right).
\end{aligned} \tag{4.23}$$

Otherwise, if $s = \alpha + \theta > 2$, by (4.19), we derive, for $0 < \delta < 1$, $0 < \gamma < \alpha\delta/(1+\delta)$,

$$\begin{aligned}
I_2 &\leqslant \mathbb{P}\left[\sum_{i=1}^{\lfloor t^{1+\delta}\rfloor} X_i > \delta t\right] + \mathbb{P}\left[N > t^{1+\delta}\right] \\
&= \sum_{n=1}^{\lfloor t^{1+\delta}\rfloor} \mathbb{P}[N=n]\mathbb{P}\left[\sum_{i=1}^{n} X_i > \delta t\right] + O\left(\frac{1}{t^{(1+\delta)(\alpha-\gamma)}}\right) \\
&\leqslant \frac{c\mathbb{E}[N]}{(\delta t)^{\alpha+\theta}} + \exp\left(-\frac{\delta^2 t^{1-\delta}}{c}\right) + O\left(\frac{1}{t^{(1+\delta)(\alpha-\gamma)}}\right),
\end{aligned}$$

which implies, for some $\nu > 0$,

$$I_2 = O\left(\frac{1}{t^{\alpha+\nu}}\right). \tag{4.24}$$

Now, we study the case when $0 < \alpha \leqslant 1$. For $1 < s \triangleq 1+\theta \leqslant 2, \theta > 0$, recalling (4.22) and noting that $\sum_{n=1}^{\lfloor t^\zeta\rfloor} n\mathbb{P}[N=n] \leqslant Ht^{\zeta(1-\alpha+\sigma)}$ for $\alpha > \theta > \sigma > 0, (\theta+1)/(\sigma+1) > \zeta > 1$, $H > 0$, we obtain, for some $\nu > 0$,

$$\begin{aligned}
I_2 &\leqslant \sum_{n=1}^{\lfloor t^\zeta\rfloor} \mathbb{P}[N=n]\left(n\mathbb{P}[X_1 > \delta t/2] + \frac{2^{s-1}cn}{\delta^s t^s}\right) + \mathbb{P}\left[N > t^\zeta\right] \\
&\leqslant Ht^{\zeta(1-\alpha+\sigma)}\left(\frac{\mathbb{E}\left[X_1^{1+\theta}\right]}{(\delta t/2)^{1+\theta}} + \frac{2^{s-1}c}{\delta^s t^{1+\theta}}\right) + \mathbb{P}\left[N > t^\zeta\right] \\
&= O\left(\frac{1}{t^{\alpha+\nu}}\right),
\end{aligned}$$

which, in conjunction with (4.23) and (4.24), yields, for some $\nu > 0$,

$$I_2 = O\left(\frac{1}{t^{\alpha+\nu}}\right). \tag{4.25}$$

Next, we study I_1. It is easy to obtain, for $\epsilon > 0$,

$$\begin{aligned}
I_1 &\leqslant \mathbb{P}\left[\sum_{i=1}^{\frac{(1-2\delta)t}{\mathbb{E}[A+U](1+\delta)}} (A_i \wedge (\epsilon t) + \mathbb{E}[U]) > (1-\delta)t\right] + \mathbb{P}\left[N > \frac{(1-2\delta)t}{\mathbb{E}[A+U](1+\delta)}\right] + \mathbb{P}[L > \epsilon t] \\
&\triangleq I_{11} + I_{12} + I_{13}.
\end{aligned} \tag{4.26}$$

By recalling Theorem 2.1, we know

$$\mathbb{P}\left[N > \frac{(1-2\delta)t}{\mathbb{E}[A+U](1+\delta)}\right] \sim \frac{\Gamma(\alpha+1)(\mathbb{E}[U+A](1+\delta))^\alpha}{\Phi((1-2\delta)t)}. \tag{4.27}$$

The same argument for (4.21) implies

$$I_{13} = O\left(\frac{1}{t^{\alpha+\theta}}\right). \tag{4.28}$$

Furthermore, I_{11} is upper bounded by

$$\mathbb{P}\left[\sum_{i=1}^{\frac{(1-2\delta)t}{\mathbb{E}[A+U](1+\delta)}}(A_i\wedge(\epsilon t)+\mathbb{E}[U])-(1+\delta)\mathbb{E}[A+U]\frac{(1-2\delta)t}{\mathbb{E}[A+U](1+\delta)}>\delta t\right]$$
$$\leqslant \mathbb{P}\left[\sup_n\left\{\sum_{i=1}^{n}(A_i\wedge(\epsilon t)+\mathbb{E}[U])-n(1+\delta)\mathbb{E}[A+U]\right\}>\delta t\right],$$

where in the preceding probability, $\sup_n\{\sum_{i=1}^{n}(A_i\wedge(\epsilon t)+\mathbb{E}[U])-n(1+\delta)\mathbb{E}[A+U]\}$ is equal in distribution to the stationary workload in a $D/GI/1$ queue with truncated service times with the stability condition $\mathbb{E}[(A\wedge(\epsilon t)+\mathbb{E}[U])]<(1+\delta)\mathbb{E}[A+U]$. Therefore, using a similar proof for Lemma 3.2 of [**?**], we can show that for any $\beta>0$, there exists $\epsilon>0$ such that

$$I_{11}=o\left(\frac{1}{t^{\beta}}\right),$$

which, in conjunction with (4.27), (4.28), (4.26), and (4.20), (4.21), (4.25), yields, by passing $\epsilon,\delta\to 0$ in (4.27),

$$\mathbb{P}\left[T>t\right]\lesssim\frac{\Gamma(\alpha+1)(\mathbb{E}[U+A])^{\alpha}}{\Phi(t)}. \tag{4.29}$$

Then, we prove the *lower bound*. It is easy to obtain, for $\delta>0$,

$$\begin{aligned}
\mathbb{P}[T>t]&=\mathbb{P}\left[\sum_{i=1}^{N-1}(U_i+A_i)+L>t\right]\\
&\geqslant\mathbb{P}\left[\sum_{i=1}^{N-1}(U_i+A_i)>t,N\geqslant\frac{t(1+\delta)}{\mathbb{E}[U+A]}+1\right]\\
&\geqslant\mathbb{P}\left[N\geqslant\frac{t(1+\delta)}{\mathbb{E}[U+A]}+1\right]-\mathbb{P}\left[\sum_{i=1}^{N-1}(U_i+A_i)\leqslant t,N\geqslant\frac{t(1+\delta)}{\mathbb{E}[U+A]}+1\right]\\
&\triangleq I_1-I_2.
\end{aligned}\tag{4.30}$$

For I_2, by defining $Y_i\triangleq U_i+A_i-\mathbb{E}[U+A]$, we obtain

$$I_2\leqslant\mathbb{P}\left[\sum_{i\leqslant t(1+\delta)/\mathbb{E}[U+A]}(U_i+A_i)\leqslant t\right]=\mathbb{P}\left[\sum_{i\leqslant t(1+\delta)/\mathbb{E}[U+A]}(-Y_i)\geqslant\delta t\right]$$

with $(-Y_i)\leqslant\mathbb{E}[U+A]<\infty$. By Chernoff bound, there exists $h,\eta>0$, such that

$$I_2\leqslant O\left(he^{-\eta t}\right), \tag{4.31}$$

which, by Theorem 2.1, equation (4.30) and passing $\delta\to 0$, yields

$$\mathbb{P}\left[T>t\right]\gtrsim\frac{\Gamma(\alpha+1)(\mathbb{E}[U+A])^{\alpha}}{\Phi(t)}. \tag{4.32}$$

Combining (4.29) and (4.32) completes the proof. □

4.7 Proof of Theorem 2.6

Proof: First, we prove the *upper bound*. It is easy to see that for $0 < \epsilon < 1$,

$$\begin{aligned}\mathbb{P}[T > (1+\epsilon)t] &= \mathbb{P}\left[\sum_{i=1}^{N-1}((A_i \wedge L) + U_i) + L > (1+\epsilon)t\right] \\ &\leqslant \mathbb{P}\left[\sum_{i=1}^{\lceil t/l(t)\rceil}(A_i \wedge L) > \frac{t}{2}\right] + \mathbb{P}\left[\sum_{i=1}^{\lceil t/l(t)\rceil} U_i > \frac{t}{2}\right] + \mathbb{P}\left[N > \left\lceil \frac{t}{l(t)}\right\rceil + 1\right] \\ &\quad + \mathbb{P}[L > \epsilon t] \\ &\triangleq I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.33}$$

Now, since $l(\cdot)$ is slowly varying and $\mathbb{P}[L > x] = O(\Phi(x)^{-1})$, we obtain,

$$I_4 = \mathbb{P}[L > t] = o\left(\Phi(t)^{1-\epsilon}\right). \tag{4.34}$$

By Theorem 2.2, we obtain

$$\lim_{t\to\infty} \frac{\log\left(\mathbb{P}\left[N > \left\lceil \frac{t}{l(t)}\right\rceil + 1\right]^{-1}\right)}{\log \Phi\left(\frac{t}{l(t)}\right)} = 1,$$

which, by (2.19), yields

$$\lim_{t\to\infty} \frac{\log\left(\mathbb{P}\left[N > \left\lceil \frac{t}{l(t)}\right\rceil + 1\right]^{-1}\right)}{\log\left(\Phi(t)\right)} = 1. \tag{4.35}$$

Next, we evaluate I_1 and I_2. For I_2,

$$\begin{aligned} I_2 &= \mathbb{P}\left[\sum_{i=1}^{\lceil t/l(t)\rceil} U_i > \frac{t}{2}\right] \\ &\leqslant \left\lceil \frac{t}{l(t)}\right\rceil \mathbb{P}\left[U_1 > \frac{t}{l(t)}\right] + \mathbb{P}\left[\sum_{i=1}^{\lceil t/l(t)\rceil} U_i \wedge \frac{t}{l(t)} > \frac{t}{2}\right] \\ &\triangleq I_{21} + I_{22}. \end{aligned} \tag{4.36}$$

For $\delta > 0$ and large t, due to condition (2.43) we obtain $l(t/l(t)) \geqslant (1-\delta/2)l(t)$, which yields

$$I_{21} \leqslant O\left(e^{-(1+\delta)l\left(\frac{t}{l(t)}\right)}\right) \leqslant O\left(e^{-(1+\delta)(1-\frac{\delta}{2})l(t)}\right) = o\left(\Phi(t)^{-1}\right). \tag{4.37}$$

Then, by using Chernoff bound, for $h > 0$, we obtain

$$\begin{aligned} I_{22} &= \mathbb{P}\left[e^{h\left(\sum_{i=1}^{\lceil t/l(t)\rceil} X_i \wedge \frac{t}{l(t)}\right)} > e^{ht/2}\right] \\ &\leqslant e^{-\frac{ht}{2}}\left(\mathbb{E}\left[e^{h\left(X_i \wedge \frac{t}{l(t)}\right)}\right]\right)^{\frac{t}{l(t)}+1}, \end{aligned}$$

which, by selecting $h = 4l(t)/t$, and noting that

$$e^{h\left(X_i \wedge \frac{t}{l(t)}\right)} \leqslant 1 + (e^4 - 1)\frac{l(t)}{t}\left(X_1 \wedge \frac{t}{l(t)}\right),$$

implies

$$
\begin{aligned}
I_{22} &\leqslant e^{-2l(t)} \left(\mathbb{E}\left[1 + (e^4 - 1)\frac{l(t)}{t}\left(X_1 \wedge \frac{t}{l(t)} \right) \right] \right)^{\frac{t}{l(t)}+1} \\
&\leqslant e^{-2l(t)} \left(1 + (e^4 - 1)\frac{l(t)}{t}\mathbb{E}\left[X_1\right] \right)^{\frac{t}{l(t)}+1} \\
&= o\left(\frac{1}{\Phi(t)} \right). \qquad (4.38)
\end{aligned}
$$

Combining (4.36), (4.37) and (4.38) yields

$$
I_2 = o\left(\frac{1}{\Phi(t)} \right). \qquad (4.39)
$$

For I_1, it is easy to see

$$
\begin{aligned}
I_1 &= \mathbb{P}\left[\sum_{i=1}^{\lceil t/l(t) \rceil} (A_i \wedge L) > \frac{t}{2} \right] \\
&\leqslant \mathbb{P}\left[\sum_{i=1}^{\lceil t/l(t) \rceil} \left(A_i \wedge \frac{t}{l(t)} \right) > \frac{t}{l(t)} \right] + \mathbb{P}\left[L > \frac{t}{l(t)} \right] \\
&\triangleq I_{11} + I_{12}. \qquad (4.40)
\end{aligned}
$$

Using the same argument as in deriving (4.38), we can prove that $I_{11} = o(1/\Phi(t))$, which, by noting condition (2.43) implying $I_{12} = o(1/\Phi(t))$, yields

$$
I_1 = o\left(\frac{1}{\Phi(t)} \right). \qquad (4.41)
$$

Combining (4.33), (4.34), (4.35), (4.39) and (4.41), yields

$$
\overline{\lim_{t \to \infty}} \frac{\log \mathbb{P}[T > t]}{\log(\Phi(t))} \leqslant -1. \qquad (4.42)
$$

Next, we prove the *lower bound*. Observe

$$
\begin{aligned}
\mathbb{P}\left[\sum_{i=1}^{N-1} (A_i + U_i) + L > t \right] &\geqslant \mathbb{P}\left[\sum_{i=0}^{N-1} (A_i \wedge 1) > t, N > \left\lceil \frac{2t}{\mathbb{E}[A \wedge 1]} \right\rceil + 1 \right] \\
&\geqslant \mathbb{P}\left[N > \left\lceil \frac{2t}{\mathbb{E}[A \wedge 1]} \right\rceil + 1 \right] - \mathbb{P}\left[\sum_{i=1}^{\left\lceil \frac{2t}{\mathbb{E}[A \wedge 1]} \right\rceil} (A_i \wedge 1) \leqslant t \right],
\end{aligned}
$$

and, by using the same arguments as in deriving (4.31), it is very easy to prove that the second probability on the right hand side of the second inequality above is exponentially bounded. Therefore, using Theorem 2.2 and the preceding exponential bound yields

$$
\underline{\lim_{t \to \infty}} \frac{\log \mathbb{P}[T > t]}{\Phi(\log t)} \geqslant -1. \qquad (4.43)
$$

Combining (4.42) and (4.43) completes the proof. □

4.8 Proof of Theorem 2.7

Proof: First, we prove the upper bound. It is easy to see that, for $\eta \triangleq \mathbb{E}[U]/\mathbb{E}[A+U]$ and $0 < \epsilon < 1$,

$$\begin{aligned}
\mathbb{P}[T > (1+\epsilon)t] &= \mathbb{P}\left[\sum_{i=1}^{N-1}((A_i \wedge L) + U_i) + L > t\right] \\
&\leqslant \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor}(A_i \wedge L) > (1-\eta)t\right] + \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor} U_i > \eta t\right] \\
&\quad + \mathbb{P}\left[N > \left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor\right] + \mathbb{P}[L > \epsilon t] \\
&\triangleq I_1 + I_2 + I_3 + I_4. \qquad (4.44)
\end{aligned}$$

The condition on L implies

$$I_4 = \mathbb{P}[L > \epsilon t] = o\left(e^{-(\log \Phi(x))^{1/(\beta+1)}}\right), \qquad (4.45)$$

and, by Theorem 2.3, we obtain

$$\lim_{t\to\infty} \frac{\log\left(\mathbb{P}\left[N > \left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[X_1]} \right\rfloor\right]^{-1}\right)}{(\log \Phi(t))^{\frac{1}{\beta+1}}} = (1-\epsilon)^{\frac{\beta}{\beta+1}} \frac{\beta^{\frac{1}{\beta+1}} + \beta^{-\frac{\beta}{\beta+1}}}{(\mathbb{E}[A+U])^{\frac{\beta}{\beta+1}}}. \qquad (4.46)$$

Now, we evaluate I_2. By applying the large deviation result proved in Theorem 3.2 (ii) of [?], and noting $\mathbb{P}[U > x] \leqslant o\left(e^{-x^{(1+\delta/2)\beta/(\beta+1)}}\right)$, we can prove that there exist $1 > \gamma > 0$ and $C > 0$, such that

$$\begin{aligned}
\mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor}(U_i \wedge \gamma\epsilon\eta t) - \eta(1-\epsilon)t > \epsilon\eta t\right] &\leqslant C\left(e^{-(\epsilon\eta t)^{(1+\delta/2))\beta/(\beta+1)}}\right) \\
&= o\left(e^{-(\log \Phi(x))^{1/(\beta+1)}}\right). \qquad (4.47)
\end{aligned}$$

Thus, considering I_2, we obtain

$$I_2 \leqslant \left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[X_1]} \right\rfloor \mathbb{P}\left[U_1 > (\gamma\epsilon\eta)t\right] + \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[X_1]} \right\rfloor}(U_i \wedge \gamma\epsilon\eta t) > \eta t\right], \qquad (4.48)$$

which, by (4.47) and the assumption on U, yields,

$$I_2 = o\left(e^{-(\log \Phi(x))^{1/(\beta+1)}}\right). \qquad (4.49)$$

For I_1, we begin with proving the situation when $\zeta = 0$, $\xi > \beta$, i.e., assuming no conditions

on $\mathbb{P}[A > x]$ beyond $\mathbb{E}[A] < \infty$. It is easy to obtain, for $0 < \epsilon < 1/(\beta+1)$,

$$
\begin{aligned}
I_1 &= \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor} (A_i \wedge L) > (1-\eta)t\right] \\
&\leqslant \mathbb{P}\left[L > t^{\frac{1}{\beta+1}-\epsilon}\right] + \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor} \left(A_i \wedge t^{\frac{1}{\beta+1}-\epsilon}\right) > (1-\eta)t\right] \\
&\triangleq I_{11} + I_{12}. \qquad (4.50)
\end{aligned}
$$

The condition $\xi > \beta$ implies, for $0 < \epsilon < (1-\beta/\xi)/(\beta+1)$,

$$
I_{11} \leqslant O\left(e^{-t^{\left(\frac{1}{\beta+1}-\epsilon\right)\xi}}\right) = o\left(e^{-(\log \Phi(t))^{1/(\beta+1)}}\right). \qquad (4.51)
$$

And, by using Chernoff bound, for $h > 0$, we obtain

$$
I_{12} = \mathbb{P}\left[e^{h\left(\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A_1]} \right\rfloor}\left(A_i \wedge t^{\frac{1}{\beta+1}-\epsilon}\right)\right)} > e^{h(1-\eta)t}\right] \leqslant e^{-h(1-\eta)t}\left(\mathbb{E}\left[e^{h\left(A_1 \wedge t^{\left(\frac{1}{\beta+1}-\epsilon\right)}\right)}\right]\right)^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A_1]} \right\rfloor},
$$

which, by selecting $h = \epsilon(1-\eta)t^{-\left(\frac{1}{\beta+1}-\epsilon\right)}$, and using $e^x \leqslant 1 + (e^b-1)x/b$ for $0 \leqslant x \leqslant b$, yields

$$
e^{h\left(A_1 \wedge t^{\left(\frac{1}{\beta+1}-\epsilon\right)}\right)} \leqslant 1 + \frac{e^{\epsilon(1-\eta)}-1}{\epsilon(1-\eta)} h\left(A_1 \wedge t^{\left(\frac{1}{\beta+1}-\epsilon\right)}\right).
$$

Then, the preceding inequalities, for ϵ small enough such that $\epsilon(1-\eta)-(1-\epsilon)(e^{\epsilon(1-\eta)}-1) > 0$, imply

$$
\begin{aligned}
I_{12} &\leqslant e^{-\epsilon(1-\eta)t^{\frac{\beta}{\beta+1}+\epsilon}}\left(\mathbb{E}\left[1 + \frac{e^{\epsilon(1-\eta)}-1}{\epsilon(1-\eta)} h\left(A_1 \wedge t^{\left(\frac{1}{\beta+1}-\epsilon\right)}\right)\right]\right)^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A_1]} \right\rfloor} \\
&\leqslant e^{-\epsilon(1-\eta)t^{\frac{\beta}{\beta+1}+\epsilon}}\left(1 + (e^{\epsilon(1-\eta)}-1)\frac{t^{\frac{\beta}{\beta+1}+\epsilon}}{t}\mathbb{E}[A_1]\right)^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A_1]} \right\rfloor} \\
&= O\left(e^{-\left(\epsilon(1-\eta)-(1-\epsilon)(e^{\epsilon(1-\eta)}-1)\right)t^{\frac{\beta}{\beta+1}+\epsilon}}\right) = o\left(e^{-(\log \Phi(t))^{1/(\beta+1)}}\right). \qquad (4.52)
\end{aligned}
$$

Combining (4.51) and (4.52) yields $I_1 = o\left(e^{-(\log \Phi(t))^{1/(\beta+1)}}\right)$ for $\zeta = 0, \xi > \beta$.

Now, in order to prove the situation $\zeta > 0$ when $\mathbb{P}[A > x]$ is bounded by a Weibull distribution, we need to use the following lemma that is based on a minor modification of Theorem 3.2 (ii) in [**?**] (or Lemma 2 in [**?**]) that can be proved by selecting $s = \upsilon Q(u)/u, 0 < \upsilon < 1$ in (5.18) of [**?**], where $Q(u)$ is defined in [**?**].

Lemma 4.2 *If $\mathbb{P}[A > x] \leqslant He^{-x^\zeta}$, $H > 0, 1 > \zeta > 0$, then, for $x^\theta < u < \epsilon x$, $\epsilon > 0$, $1 > \theta > 0$ and $n \leqslant Hx$, there exist $C > 0, 1 > \delta > 0$, such that*

$$
\mathbb{P}\left[\sum_{i=1}^{n} A_i \wedge u - n\mathbb{E}[A] > x\right] \leqslant Ce^{-\delta u^{\zeta-1}x}.
$$

Note that the case $\zeta \geqslant 1$ is trivial since in this situation I_1 is exponentially bounded using Chernoff bound. Therefore, we only need to consider the situation $0 < \zeta < 1$. Using the union bound and the independence of $\{A_i\}$ and L, it is easy to obtain, for $0 < \epsilon < 1/(\beta+1)$,

$$\begin{aligned}
I_1 &= \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor} (A_i \wedge L) > (1-\eta)t\right] \\
&\leqslant \mathbb{P}\left[L > \epsilon t\right] + \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor} \left(A_i \wedge t^{\frac{1}{\beta+1}-\epsilon}\right) > (1-\eta)t\right] \\
&\quad + \int_{t^{\frac{1}{\beta+1}-\epsilon}}^{\epsilon t} \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor} (A_i \wedge u) > (1-\eta)t\right] d\mathbb{P}[L \leqslant u] \\
&\triangleq I_{11} + I_{12} + I_{13}.
\end{aligned} \tag{4.53}$$

From (4.45) and (4.52), we obtain

$$I_{11} + I_{12} = o\left(e^{-(\log \Phi(t))^{1/(\beta+1)}}\right). \tag{4.54}$$

Applying Lemma 4.2 yields, for $t^{1/(\beta+1)-\epsilon} \leqslant u \leqslant \epsilon t$,

$$\begin{aligned}
\mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor} (A_i \wedge u) > (1-\eta)t\right] &= \mathbb{P}\left[\sum_{i=1}^{\left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor} (A_i \wedge u) - \left\lfloor \frac{(1-\epsilon)t}{\mathbb{E}[A+U]} \right\rfloor \mathbb{E}[A] > \epsilon(1-\eta)t\right] \\
&\leqslant Ce^{-\delta\epsilon(1-\eta)u^{\zeta-1}t},
\end{aligned}$$

resulting in, for some $h > 0$,

$$\begin{aligned}
I_{13} &\leqslant \int_{t^{\frac{1}{\beta+1}-\epsilon}}^{\epsilon t} Ce^{-\delta\epsilon(1-\eta)tu^{\zeta-1}} d\mathbb{P}[L \leqslant u] \\
&\leqslant Ce^{-\delta\epsilon(1-\eta)tu^{\zeta-1}}\mathbb{P}[L > u]\Big|_{\epsilon t}^{t^{\frac{1}{\beta+1}-\epsilon}} + \int_{t^{\frac{1}{\beta+1}-\epsilon}}^{\epsilon t} He^{-u^{\xi}}Ce^{-\delta\epsilon(1-\eta)tu^{\zeta-1}}(1-\zeta)\delta\epsilon(1-\eta)tu^{\zeta-2}du \\
&\leqslant \sup_{t^{\frac{1}{\beta+1}-\epsilon} \leqslant u \leqslant \epsilon t} \left\{Ce^{-u^{\xi}-\delta\epsilon(1-\eta)tu^{\zeta-1}}\right\}\left(1 + \int_{t^{\frac{1}{\beta+1}-\epsilon}}^{\epsilon t} H(1-\zeta)\delta\epsilon(1-\eta)tu^{\zeta-2}du\right) \\
&= O\left(e^{-ht^{\xi/(\xi+1-\zeta)}}\right) = o\left(e^{-(\log \Phi(t))^{1/(\beta+1)}}\right).
\end{aligned}$$

The preceding bound on I_{13}, in conjunction with (4.54) and the proof of the case for $\zeta = 0$, implies, for all $\zeta \geqslant 0$,

$$I_1 = o\left(e^{-(\log \Phi(t))^{1/(\beta+1)}}\right). \tag{4.55}$$

Thus, combining (4.44), (4.45), (4.46), (4.49), (4.55), and passing $\epsilon \to 0$ yields

$$\lim_{t\to\infty} \frac{\log \mathbb{P}[T > t]^{-1}}{(\log \Phi(t))^{\frac{1}{\beta+1}}} \geqslant \frac{\beta^{\frac{1}{\beta+1}} + \beta^{-\frac{\beta}{\beta+1}}}{(\mathbb{E}[A+U])^{\frac{\beta}{\beta+1}}}. \tag{4.56}$$

Now, we prove the *lower bound*. Using the same argument as in deriving equation (4.30) in the proof of the lower bound for Theorem 2.5, it is easy to obtain, for $\delta > 0$,

$$\mathbb{P}[T > t] \geqslant \mathbb{P}\left[N \geqslant \frac{t(1+\delta)}{\mathbb{E}[U+A]} + 1\right] - \mathbb{P}\left[\sum_{i=1}^{N-1}(U_i + A_i) \leqslant t, N \geqslant \frac{t(1+\delta)}{\mathbb{E}[U+A]} + 1\right],$$

where the second probability on the right hand side of the preceding inequality is exponentially bounded (see (4.31)). Therefore, using Theorem 2.3 and passing $\delta \to 0$ yields

$$\lim_{t\to\infty} \frac{\log \mathbb{P}[T > t]^{-1}}{(\log \Phi(t))^{\frac{1}{\beta+1}}} \leqslant \frac{\beta^{\frac{1}{\beta+1}} + \beta^{-\frac{\beta}{\beta+1}}}{(\mathbb{E}[A+U])^{\frac{\beta}{\beta+1}}}. \tag{4.57}$$

Combining (4.56) and (4.57) completes the proof. □